Cohomology Theories for Lie Groups

Cohomology: Functor $A \longmapsto H^n(G,A)$ for

- G: Lie group (fixed)
- A: abelian Lie group (more generally: a smooth G-module)

Question: What to take for $H^*(G,A)$?

Assume (unless stated otherwise): G connected, $A = \mathfrak{a}/\Gamma$

locally convex space

discrete subgroup

Want: $H^n(G,A)$ should describe certain invariants of G that are important and computable | easy to deal with.

<u>Ex</u>:

- $H^2_{sing}(G,\mathbb{Z})$ classifies line bundles on G (topological invariance)
- $H^2_{gp}(G,A)$ classifies (abstract) central extensions

$$A \to \hat{G} \to G \quad \text{(group theoretical inv.)}$$

Reminder: Group cohomology $H^n_{gp}(G,A)$

On this slide: G arbitrary, A arbitrary (abelian)

Consider: $d^n_{gp} : \mathrm{Maps}(G^n, A) \to \mathrm{Maps}(G^{n+1}, A)$

$$d^n_{gp} f(g_0, \ldots, g_n) := f(g_1, \ldots, g_n) - f(g_0 g_1, g_2, \ldots, g_n) \pm \ldots \pm f(g_0, \ldots, g_{n-1} g_n) \pm f(g_0, \ldots, g_{n-1})$$

Define: $H^n_{gp}(G,A) := \ker(d^n_{gp}) / \operatorname{im}(d^{n-1}_{gp})$ $\quad$ (n.b.: $d^n_{gp} \circ d^{n-1}_{gp} = 0$)

Properties:

- $H^1_{gp}(G,A) = \mathrm{Hom}(G,A)$
- $H^2_{gp}(G,A) \cong$ Centr. ext. of G by A $/\sim$
- $H^n_{gp}(G,A) \cong H^n_{sing}(BG, A)$

classifying space as discrete group

Reminder: Group cohomology & central extensions

On this slide: G arbitrary, A arbitrary (abelian)

$$\cong$$
$$H^2_{gp}(G,A)$$

- $f: G\times G \to A$ in $\ker(d^2_{gp}) \iff f(h,k) - f(gh,k) + f(g,hk) - f(g,h) = 0 \quad \forall g,h,k$

  $\rightsquigarrow$ consider $A\times G$, endowed with

$$(a,g)\circ(b,h) = (a+b+f(g,h),\, g\cdot h)$$

easy calculation shows: $\circ$ is associative $\iff$ $(*)$ holds

- on the other hand, if $A \longrightarrow \widehat{G} \xrightarrow{q} G$ is a central extension, then choose a section $\sigma: G\to\widehat{G}$ of q

$$\Rightarrow (g,h)\mapsto \sigma(g)\cdot\sigma(h)\cdot\sigma(gh)^{-1} \underbrace{\text{takes values in } A=\ker(q)}_{\text{since } q\circ\sigma = \mathrm{id}_G} \text{ and is in } \ker(d^2_{gp})$$

Group Theory vs. Topology I:

Problem: topological (in part. homotopic) structure is (in general) incompatible with algebraic structure!

Illustration: Heisenberg group $\mathbb{R} \xrightarrow{c \mapsto \begin{pmatrix} 1 & 0 & c \\ & 1 & 0 \\ & & 1 \end{pmatrix}} \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \longmapsto (a,b)} \mathbb{R} \times \mathbb{R}$

is a non-trivial (in part. non-abelian) central extension of $\mathbb{R}\times\mathbb{R}$ by $\mathbb{R}$.

But: $H^n_{sing}(B\mathbb{R}, \mathbb{R}) = 0$ (for $n \geq 1$), since $\mathbb{R}$ is contractible ($\Rightarrow B\mathbb{R} \simeq *$)

$\rightsquigarrow$ cannot take $H^n_{sing}(BG, A)$ (or even $H^n_{sheaf}(BG, \underline{A})$) as generalisation of $H^n_{gp}(G, A)$.

(Note, however, that for G compact we have $H^n_{dR}(G, \mathbb{R}) \cong H^n_{Lie}(\mathfrak{g}, \mathbb{R})$, so the topology might know something of the algebraic structure)

Group Theory vs. Topology II:

First (naive) guess for a good cohom. theory: globally smooth $H^n_{glob}(G,A)$

Set $d^n_{glob} :=$ restriction of d^n_{gp} to $d^n_{glob}: C^\infty(G^n, A) \to C^\infty(G^{n+1}, A)$

and $H^n_{glob}(G,A) := \ker(d^n_{glob}) / \operatorname{im}(d^{n-1}_{glob})$

Facts: – $H^1_{glob}(G,A) = \operatorname{Hom}_{Lie}(G,A)$

– $H^2_{glob}(G,A)$ classifies central extensions $A \to \widehat{G} \to G$ with a smooth global section (i.e. which are "topologically trivial").

– have long exact sequences for top. trivial sequences $A_1 \to A_2 \to A_3$ (in particular: not for $\Gamma \longrightarrow \mathfrak{a} \longrightarrow A$)

Moreover: $H^n_{gp}(G,A)$ vanishes for compact $\underbrace{G}^{\text{1-connected}}$ and $n \geq 1$ [Hu, van Est, Mostow].

From globally smooth to locally smooth

Topologically non-trivial central extensions

- projective groups $Z(G) \hookrightarrow G \to G/Z(G)$ (likely)
- universal coverings $\pi_1(G) \to \widetilde{G} \to G$ (always)
- Kac-Moody groups $U(1) \to \widehat{\Omega K} \to \Omega K$ (always for K compact & simple)

Note: differentiation in identity elt. gives a map $H^n_{glob}(G,A) \to H^n_{Lie}(\mathfrak{g},\mathfrak{a})$, which has proven to be very useful [van Est].

Next try: set $d^n_{loc} :=$ restriction of d^n_{gp} to

$$C^n_{loc} := \{ f \in Map(G^n, A) : f \text{ is smooth on some identity neighbourhood} \}$$

and define $H^n_{loc}(G,A) := \ker(d^n_{loc}) / \operatorname{im}(d^{n-1}_{loc})$ [Tuynman-Wiegerinck, Weinstein-Xu, Neeb]

Locally smooth group cohomology:

Def.: $H^n_{loc}(G,A) := \ker(d^n_{loc}) / \operatorname{im}(d^{n-1}_{loc})$ for d^n_{loc}: restriction of d^n_{gp} to

$$C^n_{loc}(G,A) := \{ f \in \mathrm{Map}(G^n, A) : f \text{ is smooth on some identity neighbourh.} \}$$

Fact 1: $D : H^n_{loc}(G,A) \to H^n_{Lie}(\mathfrak{g}, \mathfrak{a})$ differentiation in identity element

Fact 2: $H^2_{loc}(G,A)$ classifies central extensions $A \to \widehat{G} \to G$, having a local section

Fact 3: Existence of long exact sequences in coefficients (i.e. for $\Gamma \to \mathfrak{a} \to A$)

Fact 4 [Neeb]: $\exists$ an exact sequence $\cdots \to \mathrm{Hom}(\pi_2(BG), A) \to H^2_{loc}(G,A) \xrightarrow{D} H^2_{Lie}(\mathfrak{g}, \mathfrak{a}) \to \cdots$

Fact 5: G paracompact $\Rightarrow H^2_{loc}(G,A) \cong H^2_{glob}(G,A)$ for contractible A

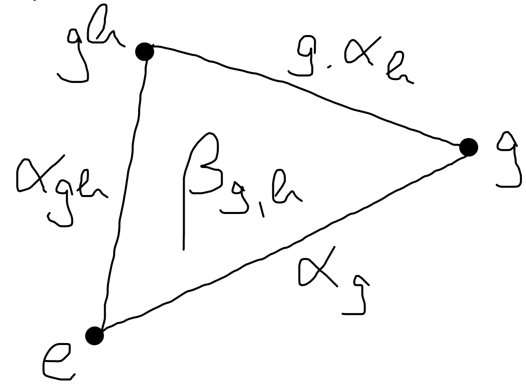
Example: The string cocycle

(for simple, compact and 1-connected)

choose: $\forall g \in G$ a smooth path

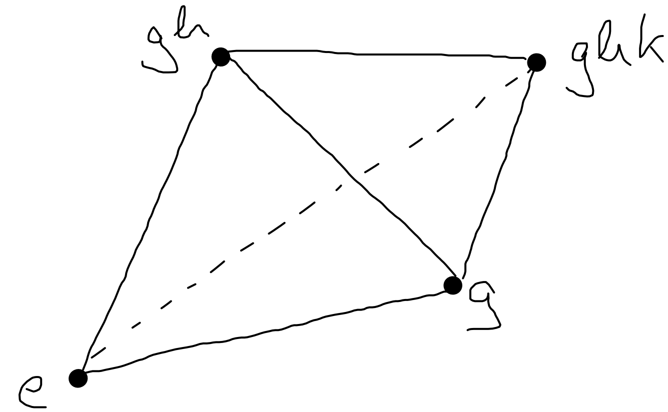
(more precisely: $\alpha : G \to P_e G$ a locally smooth section)

$\forall g,h \in G$ a smooth map $\beta_{g,h} : \Delta^2 \longrightarrow G$ such that



i.e. $\partial \beta_{g,h} = g.\alpha_h - \alpha_{gh} + \alpha_g$

$\forall g,h,k \in G$ choose a "filler" $\gamma_{g,h,k}$ of



i.e. $\gamma_{g,h,k} : \Delta^3 \longrightarrow G$ with

$$\partial \gamma_{g,h,k} = g.\beta_{h,k} - \beta_{gh,k} + \beta_{g,hk} - \beta_{g,h}$$

If now $\omega = \langle [\cdot,\cdot],\cdot \rangle^e$ is the canonical 3-Form on G, then $G^3 \ni (g,h,k) \longmapsto \exp\left(2\pi i \int_{\gamma_{g,h,k}} \omega \right) \in U(1)$

is a locally smooth 3-cocycle on G, the "string cocycle".

(Corresponding categorified Lie groups show up in higher gauge theory and elliptic cohomology.)

Further examples:

·) universal classes $[\vartheta_n] \in H^{n+1}_{loc}(G, \pi_n(G))$ (for G $(n-1)$-connected, generalize simply conn. cover)

·) locally smooth cocycles from Godbillion-Vey classes ($\rightsquigarrow$ higher analoga of Virasoro groups)

$\rightsquigarrow$ All this asks for a conceptual approach to $H^*_{loc}(G, A)$. In particular:

- a conceptual definition would be nice !
- what are interpretations (algebraic/geometric) for $H^n_{loc}(G, A)$ for $n \geq 3$?
- calculability of $H^*_{loc}(G, A)$? (recall: easy to define $\Leftrightarrow$ hard to compute)
- comparison of H^*_{loc} and H^*_{glob} ?

Locally smooth vs. globally smooth

Recall: $H^n_{glob}(G,\mathfrak{a}) \xrightarrow{\varphi} H^n_{loc}(G,\mathfrak{a})$ is an iso for $n=1,2$, G paracompact

(proof is rather elementary, only uses sheaf / Čech cohomology)

Thm [Fuchssteiner-W.]: φ is an iso for all n (and Koeff. $\mathfrak{a}$).

Proof (idea): – Fuchssteiner has developed various spectral sequences in his PHD thesis, encoding the extendability of locally defined cocycles.

– The obstructions for extending a locally defined cocycle are given in terms of (abstract) Alexander-Spanier cohomology.

– For contractible coeff., abstract and smooth Alexander-Spanier cohomology agree.

$\Rightarrow$ obstructions for abstract extendibility and smooth extendibility are the same! $\square$

The conceptual approach to $H^n_{loc}(G,A)$

$H^*_{loc}(G,A)$ as a derived functor (assume G to be paracompact):

Slogan: simplicial theory captures algebraic and homotopic aspects!

So: treat G as simplicial space $BG_\bullet$ with

·) $BG_n := \underbrace{G \times \_\_ \times G}_{n\text{-times}}$ ⟿ algebraic structure

·) $d^j_n(g_0,\ldots,g_n) = (g_0,\ldots,g_{j-1}\cdot g_j,\ldots,g_n)$, $s^j_n(g_0,\ldots,g_n) = (g_0,\ldots,g_{j-1},e,g_j,\ldots,g_n)$

Note: $|BG_\bullet|$ can be taken as a model for BG, whence the name ⟿ top. structure

Benefit: for simplicial spaces there exists a sheaf theory (like for top. spaces):

Def: A sheaf (of ab. groups) on $BG_\bullet$ is a collection of sheaves on each BG_n, compatible with the structure maps d^j_n and s^j_n

Simplicial sheaf cohomology I

Facts [Grothendieck, Deligne, Friedlander]:

- The category of sheafs on an arbitrary simpl. space is abelian
- There exists the notion of a section functor Γ
- Γ is left-exact

$\rightsquigarrow$ can define $H^n_{simpl}(BG_\bullet, \underline{A})$ as derived functors of Γ !

- If A is discrete abelian, then $H^n_{simpl}(BG_\bullet, \underline{A}) \cong H^n_{sing}(|BG_\bullet|, A)$ [Deligne]

Key point: If the sheaves on each BG_n $(=G^n)$ have "no cohomology", then we may compute $H^n_{simpl}(BG_\bullet, \underline{A})$ "as if it were group cohom."

Simplicial sheaf cohomology II

Key point: If the sheaves on each $BG_n (= G^n)$ have "no cohomology", then we may compute $H^n_{simpl}(BG_\bullet, \underline{A})$ "as if it were group cohom."

Corollary: $H^n_{simpl}(BG_\bullet, \underline{\mathfrak{a}}) \cong H^n_{glob}(G, \mathfrak{a})$, since $H^n_{sheaf}(X, \mathfrak{a})$ vanishes (for $n \geq 1$, X paracompact)

Corollary: G compact $\Rightarrow H^n_{simpl}(BG_\bullet, \underline{A}) \cong H_{sing}(BG, \Gamma)$ (recall: $A = \mathfrak{a}/\Gamma$)

Proof: Consider the exact sequence $\underline{\Gamma} \longrightarrow \underline{\mathfrak{a}} \longrightarrow \underline{A} \Longrightarrow$ exact seq.

$$
\begin{array}{ccccccc}
H^n_{simpl}(BG_\bullet, \underline{\mathfrak{a}}) \longrightarrow & H^n_{sing}(BG_\bullet, \underline{A}) \longrightarrow & H^n_{simpl}(BG_\bullet, \underline{\Gamma}) \longrightarrow & H^{n-1}_{simpl}(BG_\bullet, \underline{\mathfrak{a}}) \\
\cong & & \cong & \cong \\
H^n_{glob}(G, \mathfrak{a}) = 0 & & H^n_{sing}(|BG_\bullet|, \Gamma) & H^{n-1}_{glob}(G, \mathfrak{a}) = 0
\end{array}
$$

$\square$

$\rightsquigarrow$ Question: Relation between $H^n_{simpl}(BG_\bullet, A)$ and $H^n_{loc}(G, A)$?

Locally smooth vs. simplicial cohomology

For a pointed manifold $(*,X)$ consider the sheaf $\underline{A}_{loc}$, given by

$$U \mapsto \underline{A}_{loc}(U) = \begin{cases} \mathrm{Map}(U,A) & \text{if } * \notin U \\ \{f \in \mathrm{Map}(U,A) : f \text{ is smooth on some neighb. of } x\} & \text{if } * \in U \end{cases}$$

<u>Fact</u>: $\underline{A}_{loc}$ is a soft sheaf for smoothly paracompact $X \Longrightarrow H^n_{sheaf}(X, \underline{A}_{loc}) = 0$ (for $n \geq 1$)

$\Longrightarrow H^n_{simpl}(BG_{\bullet}, \underline{A}_{loc}) \cong H^n_{loc}(G,A)$ for smoothly paracompact G.

<u>Note</u>: There is a canonical morph. of (simplicial) sheafs $\underline{A} \longrightarrow \underline{A}_{loc}$

$\Longrightarrow$ There exists an induced morph. $H^n_{simpl}(BG_{\bullet}, \underline{A}) \xrightarrow{\psi} H^n_{simpl}(BG_{\bullet}, \underline{A}_{loc}) \cong H^n_{loc}(G,A)$

Thm [Wagemann-W., Schommer-Pries]: Replacing "(locally) smooth" by "(locally) continuous", ψ is an isomorphism. (Smooth case maybe similarly).

Interpretation of $H^n_{simpl}(BG, \underline{A})$

Thm [Schommer-Pries]: $H^3_{simpl}(BG, \underline{A})$ classifies central extensions

$$\left({}^{*}/\!\!/_{A}\right) \longrightarrow \mathcal{E} \longrightarrow \underline{G}$$

← result uses simpl. Čech cohomology

of smooth group stacks (for G fin. dim.)

(Occurs in his investigation of a finite-dimensional String 2-group.)

Note [Schreiber]: $H^n_{simpl}(BG, \underline{A})$ may be described as intrinsic cohomology of an ∞-topos!

Moreover: There exist many other approaches to define suitable cohom. groups of Lie groups (Wigner, Borel, Segal), which all turned out to be $H^n_{simpl}(BG, \underline{A})$