Infinite-Dimensional Lie Theory for Gauge Groups

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To my parents

for teaching me poems and fishes

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Abstract

Ziel dieser Arbeit ist die Initialisierung einer Lie-theoretischen Behandlung von Symmetriegruppen von Hauptfaserbündeln, hauptsächlich von Eichgruppen. Für ein fixes K-Hauptfaserbündel $\mathcal{P}$ bezeichnen wir diese mit $\mathrm{Gau}(\mathcal{P})$ und identifizieren sie meistens mit der Gruppen der äquivarianten glatten Abbildungen $C^{\infty}(P,K)^{K}$. Diese Gruppen werden als unendlichdimensionale lokalkonvexe Lie-Gruppen behandelt. Da unendlichdimensionale Lie-Theorie ein Gebiet ist, das momentan einem regen Forschugsprozess unterworfen ist und die Terminologie noch nicht gefestigt ist, müssen wir die Fragestellung präzisieren. In dieser Arbeit wird den folgenden Fragen nachgegangen:

- Für welche $\mathcal{P}$ ist $\mathrm{Gau}(\mathcal{P})$ eine unendlichdimensionale Lie-Gruppe?
- Wie können die Homotopiegruppen $\pi_n(\mathrm{Gau}(\mathcal{P}))$ bestimmt werden?
- Wie sieht die Erweiterungstheorie von $\mathrm{Gau}(\mathcal{P})$ aus?

Dies ist natürlich nur ein kleiner Teil der Fragen, die mit Lie-Gruppen verbunden sind. Sie können alle mit der gleichen Idee behandelt werden, die wir im Folgenden beschreiben. Ein Bündel kann (bis auf Äquivalenz) auf mehrere verschiedenen Arten beschrieben werden. Zwei verschiedene Arten sind durch die Beschreibung durch eine klassifizierende Abbildung $f_{\mathcal{P}}$ und durch einen Kozyklus $\mathcal{K}_{\mathcal{P}}$ gegeben. Eine klassifizierende Abbildung $f_{\mathcal{P}}$ ist eine global definierte Abbildung mit Werten in einem klassifizierenden Raum, während ein Kozyklus aus vielen lokal definierten Abbildungen besteht, die Werte in der Lie-Gruppe K annehmen und bestimmte Kompatibilitätsbedingungen erfüllen. Diese beiden Objekte, klassifizierende Abbildungen und Kozyklen, leben in zwei verschiedenen Welten, nämlich Topologie und Lie-Theorie.

Die Idee ist nun, diese beiden Konzepte zu kombinieren und die bestehenden Resultate aus Topologie und Lie-Theorie zu benutzen um Antworten auf die oben genannten Fragen zu erhalten. Da diese Fragen recht allgemein gehalten sind kann man nicht erwarten, Antworten in dieser Allgemeinheit zu erhalten. In dieser Arbeit werden wir jedoch viele interessante Fälle aus der mathematischen Physik behandeln. Die dabei erzielten Resultate beinhalten:

- Konstruktion einer Lie-Gruppenstruktur auf $\mathrm{Gau}(\mathcal{P})$ falls die Strukturgruppe lokal exponentiell ist.
- Eine kanonische schwache Homotopieäquivalenz $\mathrm{Gau}(\mathcal{P}) \to \mathrm{Gau}_c(\mathcal{P})$.
- Entwicklung eines Gättungsverfahrens für Hauptfaserbündel.
- Konstruktion einer Erweiterung $\mathrm{Gau}(\mathcal{P}) \to \mathrm{Aut}(\mathcal{P}) \to \mathrm{Diff}(M)_{\mathcal{P}}$.
- Bestimmung einiger Homotopiegruppen und aller rationalen Homotopiegruppen von $\mathrm{Gau}(\mathcal{P})$ für endlichdimensionale Hauptfaserbündel über Sphären.
- Konstruktion zentraler Erweiterungen $Z \to G_{\mathcal{P}} \to \mathrm{Gau}(\mathcal{P})_0$.
- Konstruktion einer automorphen Wirkung von $\mathrm{Aut}(\mathcal{P})$ auf $G_{\mathcal{P}}$.
- Anwendung auf affine getwistete Kac–Moody Gruppen.

Abstract

The aim of this thesis is to consider symmetry groups of principal bundles and to initiate a Lie theoretic treatment of these groups. These groups of main interest are called gauge groups. When taking a particular principal K-bundle $\mathcal{P}$ into account, we denote the gauge group of this bundle by $\mathrm{Gau}(\mathcal{P})$, which we mostly identify with the space of smooth K-equivariant mappings $C^\infty(P,K)^K$. These groups will be treated as infinite-dimensional Lie groups, modelled on an appropriate vector space. Since Lie theory in infinite dimensions is a research area which is presently under active development, this terminology is not settled, and we have to make precise what we mean with "infinite-dimensional Lie theory". The following questions are considered in this thesis:

- For which bundles $\mathcal{P}$ is $\mathrm{Gau}(\mathcal{P})$ an infinite-dimensional Lie group, modelled on an appropriate locally convex space?
- How can the homotopy groups $\pi_n(\mathrm{Gau}(\mathcal{P}))$ be computed?
- What extensions does $\mathrm{Gau}(\mathcal{P})$ permit?

Of course, this is only a marginal part of the questions that come along with Lie groups. These problems have in common that they can be approached with the same idea, which we describe now. Along with a bundle $\mathcal{P}$ come many different ways of describing it (up to equivalence). Two fundamental different ways are given by describing $\mathcal{P}$ either in terms of a classifying map $f_{\mathcal{P}}$, or by a cocycle $\mathcal{K}_{\mathcal{P}}$. A classifying map $f_{\mathcal{P}}$ is a globally defined map $f_{\mathcal{P}}$ with values in some classifying space, while a cocycle consists of many locally defined maps, with values in a Lie group, obeying some compatibility conditions. These objects, classifying maps and cocycles, live in two different worlds, namely topology and Lie theory.

The idea now is to combine these two concepts and to use the existing tools from topology and Lie theory in order to give answers to the questions above. Since the questions are formulated quite generally, we cannot hope to get answers in full generality, but for many interesting cases occurring in mathematical physics, we will provide answers. These include:

- Construction of a Lie group structure on $\mathrm{Gau}(\mathcal{P})$ if the structure group is locally exponential.
- Showing that the canonical inclusion $\mathrm{Gau}_c(\mathcal{P}) \to \mathrm{Gau}(\mathcal{P})$ is a weak homotopy equivalence.
- Providing a smoothing procedure for continuous principal bundles.
- Construction of an Extension of Lie groups $\mathrm{Gau}(\mathcal{P}) \to \mathrm{Aut}(\mathcal{P}) \to \mathrm{Diff}(M)_{\mathcal{P}}$.
- Calculation of some homotopy groups and of all rational homotopy groups of $\mathrm{Gau}(\mathcal{P})$ for finite-dimensional principal bundles over spheres.
- Construction of central extensions $Z \to G_{\mathcal{P}} \to \mathrm{Gau}(\mathcal{P})_0$.
- Construction of an automorphic action of $\mathrm{Aut}(\mathcal{P})$ on $G_{\mathcal{P}}$.
- Applications to affine twisted Kac–Moody groups.

Contents

Chapter 1

Introduction

Bundle theory and Lie theory are two of the most important topics in Mathematical Physics. Bundles occur naturally in the description of many physical systems, often in terms of (co-) tangent bundles of manifolds or in terms of principal bundles. These descriptions always carry redundant information, emerging from introducing coordinates or from geometrical realisations. This redundant information gives rise to symmetries of the mathematical description, which can be expressed in terms of groups. In many interesting cases, these groups are geometric objects itself and are called Lie groups.

One of the most popular examples is general relativity, which is formulated in terms of manifolds and the curvature of vector bundles. The pioneering idea of Einstein was that any point and any coordinate system of the manifold should have equal physical laws. This assumption leads to a theory which is invariant under diffeomorphisms by assumption. Thus general relativity may be viewed as a theory formulated in terms of manifolds M and their tangent bundles TM, which has the Lie group $\mathrm{Diff}(M)$ as symmetry group.

The aim of this thesis is to consider symmetry groups of principal bundles and to initiate a Lie theoretic treatment of these groups. The groups of main interest are gauge groups, which can be viewed as the "internal" symmetry groups of quantum field theories (cf. [MM92] [Na00]). When taking a particular principal bundle $\mathcal{P}$ into account, we denote the gauge group of this bundle by $\mathrm{Gau}(\mathcal{P})$. These groups will be treated as infinite-dimensional Lie groups, modelled on an appropriate vector space. Since Lie theory in infinite dimensions is a research area which is presently under active development, this terminology is not settled, and we have to make precise what we mean with "infinite-dimensional Lie theory". The following questions are considered in this thesis:

- For what bundles $\mathcal{P}$ is $\mathrm{Gau}(\mathcal{P})$ an infinite-dimensional Lie group, modelled on an appropriate locally convex space?
- How can the homotopy groups $\pi_n(\mathrm{Gau}(\mathcal{P}))$ be computed?
- What extensions does $\mathrm{Gau}(\mathcal{P})$ permit?

Of course, this is only a marginal part of the questions that come along with Lie groups. These problems have in common that they can be approached with the same idea, which we describe now. Along with a bundle $\mathcal{P}$ come many different ways of describing it (up to equivalence). Two fundamental different ways are given by describing $\mathcal{P}$ either in terms of a classifying map $f_{\mathcal{P}}$, or by a cocycle $\mathcal{K}_{\mathcal{P}}$. A classifying map $f_{\mathcal{P}}$ is a globally defined map $f_{\mathcal{P}}$ with values in some classifying space, while a cocycle consists of many locally defined maps, with values in a Lie group, obeying some compatibility conditions. These objects, classifying maps and cocycles, live in two different worlds, namely topology and Lie theory.

The idea now is to combine these two concepts and to use the existing tools from topology and Lie theory in order to give answers to the questions above. Since the questions are formulated quite generally, we cannot hope to get answers in full generality, but for many interesting cases occurring in mathematical physics, we will provide answers.

We now give a rough outline of the results that can be found in this thesis, without going into too much detail. Throughout the thesis, we always assume that the base spaces of the bundles under consideration are *connected*.

Chapter 2: In the first section, we introduce manifolds with corners, which are the objects that we use extensively throughout the thesis. We have the need to work with these objects, since we are forced to consider compact subsets of certain open subsets of a manifold as manifolds themselves (e.g., $[0,1]^n$ as a manifold with corners in $\mathbb{R}^n$). Since we want to work with mapping spaces, we take a quite uncommon definition of a manifold with corners, which we show to be equivalent to the usual one later in the chapter.

In the second section, we introduce mapping spaces and topologies on them. In particular, we define the C^∞-topology on spaces of smooth mappings between manifolds, which is the topology we use throughout this thesis. Along with this, we show and recall some basic facts on spaces of smooth mappings with values in locally convex spaces or Lie groups and on spaces of smooth sections in vector bundles. These facts are the Lie theoretic tools for mapping spaces, mentioned above, which we use.

In the last section, we relate our concept of a manifold with corners to the one more frequently used in the literature. The results of this section are also well-known, but we will derive alternative proofs.

Chapter 3: In this chapter, we introduce Lie group structures on the gauge group $\mathrm{Gau}(\mathcal{P})$ and on the automorphism group $\mathrm{Aut}(\mathcal{P})$ of a principal bundle $\mathcal{P}$ over a compact manifold M. In the first section, we consider the gauge group $\mathrm{Gau}(\mathcal{P})$ and introduce a Lie group topology on it under a technical requirement. This requirement, called "property SUB", encodes exactly what we need to ensure the construction of a canonical Lie group topology on $\mathrm{Gau}(\mathcal{P})$.

Theorem (Lie group structure on Gau($\mathcal{P}$)). *Let $\mathcal{P}$ be a smooth principal K-bundle over the compact manifold M (possibly with corners). If $\mathcal{P}$ has the property SUB, then* $\mathrm{Gau}(\mathcal{P}) \cong C^{\infty}(P,K)^{K}$ *carries a Lie group structure, modelled on $C^{\infty}(P,\mathfrak{k})^{K}$. If, moreover, K is locally exponential, then* $\mathrm{Gau}(\mathcal{P})$ *is so.* ■

In the remainder of the section, we discuss the question what bundles have the property SUB. Most bundles (including all bundles modelled on Banach spaces) have this property.

In the second section, we derive a first major step towards the computation of the homotopy groups $\pi_n(\mathrm{Gau}(\mathcal{P}))$ of the gauge group. Following ideas from mapping groups, we reduce the determination of $\pi_n(\mathrm{Gau}(\mathcal{P}))$ to the case of continuous gauge transformations $\mathrm{Gau}_c(\mathcal{P})$.

Theorem (Weak homotopy equivalence for Gau($\mathcal{P}$)). *Let $\mathcal{P}$ be a smooth principal K-bundle over the compact manifold M (possibly with corners). If $\mathcal{P}$ has the property SUB, then the natural inclusion* $\iota : \mathrm{Gau}(\mathcal{P}) \hookrightarrow \mathrm{Gau}_c(\mathcal{P})$ *of smooth into continuous gauge transformations is a weak homotopy equivalence, i.e., the induced mappings* $\pi_n(\mathrm{Gau}(\mathcal{P})) \to \pi_n(\mathrm{Gau}_c(\mathcal{P}))$ *are isomorphisms of groups for $n \in \mathbb{N}_0$.* ■

This theorem is the first connection between the two worlds described above, i.e., Lie theory (considering $\mathrm{Gau}(\mathcal{P})$ as the object of interest) and topology (considering $\mathrm{Gau}_c(\mathcal{P})$ as the object of interest). It reduces the determination of $\pi_n(\mathrm{Gau}(\mathcal{P}))$ completely to the determination of $\pi_n(\mathrm{Gau}_c(\mathcal{P}))$, which we will consider in Chapter 4.

In the third section, we develop the technique of reducing problems for gauge transformations to problems on Lie group valued mappings, satisfying some compatibility conditions further, to bundle equivalences. With the aid of some technical constructions, we derive the following two theorems, which are somewhat apart from the main objective of this chapter.

Theorem (Smoothing continuous principal bundles). *Let K be a Lie group modelled on a locally convex space, M be a finite-dimensional paracompact manifold (possibly with corners) and $\mathcal{P}$ be a continuous principal K-bundle over M. Then there exists a smooth principal K-bundle $\widetilde{\mathcal{P}}$ over M and a continuous bundle equivalence $\Omega : \mathcal{P} \to \widetilde{\mathcal{P}}$.* ■

Theorem (Smoothing continuous bundle equivalences). *Let K be a Lie group modelled on a locally convex space, M be a finite-dimensional paracompact manifold (possibly with corners) and $\mathcal{P}$ and $\mathcal{P}'$ be two smooth principal K-bundles over M. If there exists a continuous bundle equivalence $\Omega : P \to P'$, then there exists a smooth bundle equivalence $\widetilde{\Omega} : P \to P'$.* ■

Again, these theorems provide an interplay between locally defined Lie group valued functions with compatibility conditions on the one hand and classifying maps in classifying spaces on the other, because the classical proof of these theorems in the case of finite-dimensional bundles uses classifying maps.

The last section of Chapter 3 is a first approach to the extension theory of $\operatorname{Gau}(\mathcal{P})$. One way of defining $\operatorname{Gau}(\mathcal{P})$ is to consider it as a normal subgroup of $\operatorname{Aut}(\mathcal{P})$, i.e., $\operatorname{Aut}(\mathcal{P})$ is the extension of some group isomorphic to $\operatorname{Aut}(\mathcal{P})/\operatorname{Gau}(\mathcal{P})$ by $\operatorname{Gau}(\mathcal{P})$. By using techniques from the Lie theory of mapping spaces, we put this into a Lie theoretic context.

Theorem ($\operatorname{Aut}(\mathcal{P})$ as an extension of $\operatorname{Diff}(M)_{\mathcal{P}}$ by $\operatorname{Gau}(\mathcal{P})$). *Let $\mathcal{P}$ be a smooth principal K-bundle over the closed compact manifold M. If $\mathcal{P}$ has the property SUB, then $\operatorname{Aut}(\mathcal{P})$ carries a Lie group structure such that we have an extension of smooth Lie groups*

$$\operatorname{Gau}(\mathcal{P}) \hookrightarrow \operatorname{Aut}(\mathcal{P}) \xrightarrow{Q} \operatorname{Diff}(M)_{\mathcal{P}},$$

where $Q : \operatorname{Aut}(\mathcal{P}) \to \operatorname{Diff}(M)$ is the canonical homomorphism and $\operatorname{Diff}(M)_{\mathcal{P}}$ is the open subgroup of $\operatorname{Diff}(M)$ preserving the equivalence class of $\mathcal{P}$ under pull-backs. ■

Chapter 4: In this chapter, we turn to the computation of $\pi_n(\operatorname{Gau}_c(\mathcal{P}))$, which we have seen to be isomorphic to $\pi_n(\operatorname{Gau}(\mathcal{P}))$ in Chapter 3. We can thus work in a purely topological setting and take the existing tools of homotopy theory into account. In the first section, we explain how the problem of the determination of $\operatorname{Gau}_c(\mathcal{P})$ can be expressed in terms of long exact homotopy sequences and connecting homomorphisms.

In the second section, we show how the connecting homomorphisms, mentioned above, can be computed in terms of homotopy invariants of the structure group and the bundle. The crucial tool will be the evaluation fibration $\operatorname{ev} : \operatorname{Gau}_c(\mathcal{P}) \to K$, determined uniquely by $p_0 \cdot \operatorname{ev}(f) = f(p_0)$ for some basepoint p_0. Furthermore, it will turn out that the case of bundles over spheres is the generic one.

Theorem (Connecting homomorphism is the Samelson product). *Let K be locally contractible and $\mathcal{P}$ be a continuous principal K-bundle over*

$\mathbb{S}^m$, represented by

$$b \in \pi_{m-1}(K) \cong [\mathbb{S}^m, BK]_* \cong \mathrm{Bun}(\mathbb{S}^m, K).$$

Then the connecting homomorphisms $\delta_n : \pi_n(K) \to \pi_{n+m-1}(K)$ in the long exact homotopy sequence

$$\cdots \to \pi_{n+1}(K) \xrightarrow{\delta_{n+1}} \pi_{n+m}(K) \to \pi_n(\mathrm{Gau}_c(\mathcal{P})) \to \pi_n(K) \to \delta_n \pi_{n+m-1}(K) \to \cdots$$

induced by the evaluation fibration, are given by $\delta_n(a) = -\langle b, a\rangle_S$, where $\langle \cdot, \cdot \rangle_S$ denotes the Samelson product. ■

In the last section of Chapter 4, we explain how this exact sequence can be used to compute $\pi_n(\mathrm{Gau}_c(\mathcal{P}))$. Since for many questions in infinite-dimensional Lie theory it suffices to know the rational homotopy groups $\pi_n^{\mathbb{Q}}(\mathrm{Gau}_c(\mathcal{P}))$, we focus on $\pi_n^{\mathbb{Q}}(\mathrm{Gau}_c(\mathcal{P}))$.

Theorem (Rational homotopy groups of gauge groups). *Let K be a finite-dimensional Lie group and $\mathcal{P}$ be a continuous principal K-bundle over X, and let Σ be a compact orientable surface of genus g. If $X = \mathbb{S}^m$, then*

$$\pi_n^{\mathbb{Q}}(\mathrm{Gau}_c(\mathcal{P})) \cong \pi_{n+m}^{\mathbb{Q}}(K) \oplus \pi_n^{\mathbb{Q}}(K)$$

for $n \geq 1$. If $X = \Sigma$ and K is connected, then

$$\pi_n^{\mathbb{Q}}(\mathrm{Gau}_c(\mathcal{P})) \cong \pi_{n+2}^{\mathbb{Q}}(K) \oplus \pi_{n+1}^{\mathbb{Q}}(K)^{2g} \oplus \pi_n^{\mathbb{Q}}(K)$$

for $n \geq 1$. ■

Since the rational homotopy groups of finite-dimensional Lie groups are known, this yields a complete description of the rational homotopy groups of gauge groups for finite-dimensional bundles with connected structure group over spheres and compact surfaces.

Chapter 5: In this chapter, we consider the construction of central extensions of $\mathrm{Gau}(\mathcal{P})$ and applications to Kac–Moody groups. In the first section, we consider the construction of a central extension of the gauge algebra $\mathfrak{g} := \mathfrak{gau}(\mathcal{P})$, which is motivated by the corresponding construction for trivial bundles. This central extension $\widehat{\mathfrak{g}}_\omega$ is given by a "covariant" cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}_M(Y)$, which is constructed with the aid of some K-invariant bilinear form $\kappa : \mathfrak{k} \times \mathfrak{k} \to Y$. The target space $\mathfrak{z}_M(Y)$ of ω is some locally convex space $\mathfrak{z}_M(Y)$, which depends on Y and on the base manifold M of the bundle $\mathcal{P}$ under consideration.

In the second and third section, we check the integrability conditions from the established theory of central extensions of infinite-dimensional Lie groups for the central extension $\widehat{\mathfrak{g}}_\omega$. We again encounter the interplay between the Lie theoretic properties of $\mathrm{Gau}(\mathcal{P})$ and the topological properties of $\mathcal{P}$, which make the proof of the following theorem work.

Theorem (Integrating the central extension of $\mathfrak{gau}(\mathcal{P})$). *Let $\mathcal{P}$ be a finite-dimensional smooth principal K-bundle over the closed compact manifold M and $\kappa : \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ be universal. Furthermore, set $\mathfrak{z} := \mathfrak{z}_M(V(\mathfrak{k}))$, $\mathfrak{g} := \mathfrak{gau}(\mathcal{P})$ and $G := \mathrm{Gau}(\mathcal{P})_0$. If $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ is the covariant cocycle, then the central extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}}_\omega \twoheadrightarrow \mathfrak{g}$ of Lie algebras integrates to an extension of Lie groups $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$.* ■

In the third section, we also consider the construction of a canonical action of the automorphism group $\mathrm{Aut}(\mathcal{P})$ of the bundle $\mathcal{P}$ on the central extension $\widehat{\mathfrak{g}}_\omega$. This action will become important in the last section, because it is closely related to Kac–Moody algebras and their automorphisms. At the end of the section, we show that we also get a canonical action of $\mathrm{Aut}(\mathcal{P})$ on the central extension $\widehat{G}$.

Theorem (Integrating the $\mathrm{Aut}(\mathcal{P})$-action on $\widehat{\mathfrak{gau}(\mathcal{P})}$). *Let $\mathcal{P}$ be a finite-dimensional smooth principal K-bundle over the closed compact manifold M and set $\mathfrak{g} := \mathfrak{gau}(\mathcal{P})$ and $G := \mathrm{Gau}(\mathcal{P})_0$. If $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ is the covariant cocycle and if $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ is the central extension from the preceding theorem, then the smooth action of $\mathrm{Aut}(\mathcal{P})$ on $\widehat{\mathfrak{g}}_\omega$ integrates to a smooth action of $\mathrm{Aut}(\mathcal{P})$ on $\widehat{G}$.* ■

As an application, we describe in the last section of Chapter 5 the relation of the results of the previous chapters to Kac–Moody groups. After making the setting of Kac–Moody groups precise, we consider in particular their homotopy groups and show how the automorphic action of $\mathrm{Aut}(\mathcal{P})$ on $\widehat{\mathfrak{g}}_\omega$ leads to a geometric description of the automorphism group of twisted loop algebras. In the end of this section we give an outlook how the results of this thesis can be used to construct generalisations of Kac–Moody algebras and groups.

The thesis is organised as follows. In the beginning of each chapter and section, we give a rough outline of our aims. During each section, we give ongoing comments that should motivate the procedure of the section and should illustrate the flow of ideas. Terminology and notation can mostly be found in remarks and definitions, as long as they are important for the sequel.

Relations of the work presented in this thesis to work of other authors (at least as long as they are known to the author of the thesis), ideas for further research and open problems can be found at the end of each section and sometimes in the motivating text at the beginning of sections and chapters. However, if we cite a result directly, we make this explicit at the point of occurrence without repeating it again at the end of the section.

In the appendix, we present some facts on infinite-dimensional Lie theory and bundle theory, which we often refer to. This presentation is not meant to be exhaustive, it should only make it easier to follow the text by stating some things explicitly instead of referring to the literature.

Chapter 2

Foundations

This chapter presents the underlying material for the following chapters. We shall introduce manifolds with corners in the first section, which we will need to consider in the topologisation of the gauge group, even for principal bundles over manifolds without boundary. The second section provides the facts on spaces of smooth maps, which we shall use in the sequel. These two concepts, manifolds with corners and spaces of smooth maps along with their properties, will be the cornerstones of the theory we will build in the following chapters. Since our definition of a manifold with corners is somewhat uncommon, we relate it to the commonly used definition of a manifold with corners in the third and last section.

2.1 Manifolds with corners

In this section we present the elementary notions of differential calculus on locally convex spaces for not necessarily open domains and introduce manifolds with corners. Since we are aiming for mapping spaces, we need a notion of differentiability involving only the values of a given function on its domain without referring to extensions of the map to some open neighbourhood.

The idea, taken from [Mi80], is to restrict attention to maps which are defined on an open and dense subset of its domain, because this determines a continuous map completely. It will turn out that with this definition, most ideas from manifolds without boundary carry over to manifolds with corners, as long as only tangent mappings and their continuity are involved.

Definition 2.1.1. Let X and Y be a locally convex spaces and $U \subseteq X$ be open. Then $f : U \to Y$ is *differentiable* or C^1 if it is continuous, for each $v \in X$ the differential quotient

$$df(x).v := \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h}$$

exists and if the map $df : U \times X \to Y$ is continuous. If $n > 1$ we inductively define f to be C^n if it is C^1 and df is C^{n-1} and to be C^∞ or *smooth* if it is C^n. We say that f is C^∞ or *smooth* if f is C^n for all $n \in \mathbb{N}_0$. We denote the corresponding spaces of maps by $C^n(U,Y)$ and $C^\infty(U,Y)$. ■

Definition 2.1.2. Let X and Y be locally convex spaces, and let $U \subseteq X$ be a set with dense interior. Then $f : U \to Y$ is *differentiable* or C^1 if it is continuous, $f_{\text{int}} := f|_{\text{int}(U)}$ is C^1 and the map

$$d\left(f_{\text{int}}\right) : \text{int}(U) \times X \to Y, \quad (x,v) \mapsto d\left(f_{\text{int}}\right)(x).v$$

extends to a continuous map on $U \times X$, which is called the *differential* df of f. If $n > 1$ we inductively define f to be C^n if it is C^1 and df is C^{n-1}. We say that f is C^∞ or *smooth* if f is C^n for all $n \in \mathbb{N}_0$. We denote the corresponding spaces of maps by $C^n(U,Y)$ and $C^\infty(U,Y)$. ■

Similarly, we introduce holomorphic mappings on non-open domains. We shall not need this concept very often.

Definition 2.1.3. If X and Y are locally convex complex vector spaces and $U \subseteq X$ has dense interior, then a smooth map $f : U \to Y$ is called *holomorphic* if f_{int} is holomorphic, i.e., if each map $df_{\text{int}}(x) : X \to Y$ is complex linear (cf. [Mi84, p. 1027]). We denote the space of all holomorphic functions on U by $\mathcal{O}(U,Y)$. ■

Remark 2.1.4. Note that in the above setting $df(x)$ is complex linear for all $x \in U$ due to the continuity of the extension of df_{int}. ■

We now introduce higher differential of smooth function, which have not been defined in Definition 2.1.2.

Remark 2.1.5. Since $\text{int}(U \times X^{n-1}) = \text{int}(U) \times X^{n-1}$ we have for $n = 1$ that $(df)_{\text{int}} = d\left(f_{\text{int}}\right)$ and we inductively obtain $(d^n f)_{\text{int}} = d^n\left(f_{\text{int}}\right)$. Hence the higher differentials $d^n f$ are defined to be the continuous extensions of the differentials $d^n(f_{\text{int}})$ and thus we have that a map $f : U \to X$ is smooth if and only if

$$d^n\left(f_{\text{int}}\right) : \text{int}(U) \times X^{n-1} \to Y$$

has a continuous extension $d^n f$ to $U \times X^{n-1}$ for all $n \in \mathbb{N}$. ■

Of course we have a chain rule, the most important tool in any notion of differential calculus. However, in the way we introduced differentiable maps we need to assume that mappings are well-behaved with respect to the interiors of the domains in order to have a chain rule.

Remark 2.1.6. If $f : U_1 \to U_2$, $g : U_2 \to Y$ with $f(\text{int}(U_1)) \subseteq \text{int}(U_2)$ are C^1, then the chain rule for locally convex spaces [Gl02a, Proposition 1.15] and $(g \circ f)_{\text{int}} = g_{\text{int}} \circ f_{\text{int}}$ imply that $g \circ f : U_1 \to Y$ is C^1 and its differential is given by $d(g \circ f)(x).v = dg(f(x)).df(x).v$. In particular, $g \circ f$ is smooth if g and f are so. ■

With the above definitions and the chain rule in mind, we can now introduce manifolds with corners, and furthermore, complex manifolds with corners.

Definition 2.1.7. (cf. [Le03] for the finite-dimensional case and [Mi80]) Let Y be a locally convex space, $\lambda_1, \ldots, \lambda_n$ be continuous linearly independent linear functionals on Y and $Y^+ := \bigcap_{k=1}^n \lambda_k^{-1}(\mathbb{R}_0^+)$. If M is a Hausdorff space, then a collection $(U_i, \varphi_i)_{i\in I}$ of homeomorphisms $\varphi_i : U_i \to \varphi(U_i)$ onto open subsets $\varphi_i(U_i)$ of Y^+ (called charts) defines a *differential structure* on M of codimension n if $\cup_{i\in I} U_i = M$ and for each pair of charts φ_i and φ_j with $U_i \cap U_j \neq \emptyset$ the *coordinate change*

$$\varphi_i\left(U_i \cap U_j\right) \ni x \mapsto \varphi_j\left(\varphi_i^{-1}(x)\right) \in \varphi_j(U_i \cap U_j)$$

is smooth in the sense of Definition 2.1.2. Furthermore, M together with a differential structure $(U_i, \varphi_i)_{i\in I}$ is called a *manifold with corners* of codimension n.

If, in addition, Y is finite-dimensional and M is paracompact, then we call M a *finite-dimensional manifold with corners*. ■

Remark 2.1.8. Note that the previous definition of a manifold with corners coincides for $Y = \mathbb{R}^n$ with the one given in [Le03] and in the case of codimension 1 and a Banach space Y with the definition of a manifold with boundary in [La99], but our notion of smoothness differs. In both cases a map f, defined on a non-open subset $U \subseteq Y$, is said to be smooth if for each point $x \in U$ there exists an open neighbourhood $V_x \subseteq Y$ of x and a smooth map f_x defined on V_x with $f = f_x$ on $U \cap V_x$. However, it will turn out that for finite-dimensional manifolds with corners the two notions coincide. ■

Definition 2.1.9 (Complex Manifold with Corners). A manifold with corners is called a *complex manifold with corners* if it is modelled on a complex vector space Y and the coordinate changes in Definition 2.1.7 are holomorphic. ■

In order to check that concepts for manifolds, which are introduced in terms of charts (e.g., the smoothness of functions) do not depend on the choice of charts, we always need the chain rule for the composition of coordinate changes. Now the chain rule (Remark 2.1.6) has an additional assumption besides the smoothness of the maps under consideration. We shall show that this assumption is always satisfied by the coordinate changes of a manifold with corners.

Lemma 2.1.10. *If M is manifold with corners modelled on the locally convex space Y and φ_i and φ_j are two charts with $U_i \cap U_j \neq \emptyset$, then $\varphi_j \circ \varphi_i^{-1}(\operatorname{int}(\varphi_i(U_i \cap U_j))) \subseteq \operatorname{int}(\varphi_j(U_i \cap U_j))$.*

Proof. Denote by $\alpha : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$, $x \mapsto \varphi_j(\varphi_i^{-1}(x))$ and $\beta = \alpha^{-1}$ the corresponding coordinate changes. We claim that $d\alpha(x) : Y \to Y$ is an isomorphism if $x \in \operatorname{int}(\varphi_i(U_i \cap U_j))$. Since β maps a neighbourhood W_x of $\alpha(x)$ into $\operatorname{int}(\varphi_i(U_i \cap U_j))$, we have $d\alpha(\beta(y)).\big(d\beta(y).v\big) = v$ for $v \in Y$ and $y \in \operatorname{int}(W_x)$ (cf.

Remark 2.1.6). Since $(y,v) \mapsto d\alpha(\beta(y)).\big(d\beta(y).v\big)$ is continuous and $\mathrm{int}(W_x)$ is dense in W_x, $d\beta(\alpha(x))$ is a continuous inverse of $d\alpha(x)$.

Now suppose $x \in \mathrm{int}(\varphi_i(U_i \cap U_j))$ and $\alpha(x) \notin \mathrm{int}(\varphi_j(U_i \cap U_j))$. Then $\lambda_i(\alpha(x)) = 0$ for some $i \in \{1,\ldots,n\}$ and thus there exists a $v \in Y$ such that $\alpha(x) + tv \in \varphi_j(U_i \cap U_j)$ for $t \in [0,1]$ and $\alpha(x) + tv \notin \varphi_j(U_i \cap U_j)$ for $t \in [-1,0)$. But then $v \notin \mathrm{im}(d\alpha(x))$, contradicting the surjectivity of $d\alpha(x)$. ■

With the aid of the invariance of interior points under coordinate changes of the preceding lemma, we now define the boundary of a manifold with corners. This should not be mixed up with the boundary for a topological space, since the latter can only be defined for topological subspaces (and the boundary of the whole space is always empty).

Remark 2.1.11. The preceding lemma shows that the points of $\mathrm{int}(Y_+)$ are invariant under coordinate changes and thus the *interior* $\mathrm{int}(M) = \bigcup_{i\in I} \varphi_i^{-1}(\mathrm{int}(Y_+))$ is an intrinsic object, attached to M. We denote by $\partial M := M \setminus \mathrm{int}(M)$ the *boundary* of M. If $\partial M = \emptyset$, i.e., if M is a manifold without boundary, then we also say that M is a *manifold without boundary* or *closed manifold* or *locally convex manifold*. ■

As indicated before, we now can say what a smooth map on a manifold with corners should be.

Definition 2.1.12. A map $f : M \to N$ between manifolds with corners is said to be C^n (respectively, *smooth*) if $f\left(\mathrm{int}(M)\right) \subseteq \mathrm{int}(N)$ and the corresponding *coordinate representation*

$$\varphi_i(U_i \cap f^{-1}(U_j)) \ni x \mapsto \varphi_j\left(f\left(\varphi_i^{-1}(x)\right)\right) \in \varphi_j(U_j)$$

is C^n (respectively, smooth) for each pair φ_i and φ_j of charts on M and N. We again denote the corresponding sets of mappings by $C^n(M,N)$ and $C^\infty(M,N)$. A smooth map $f : M \to N$ between complex manifolds with corners is said to be holomorphic if for each pair of charts on M and N the corresponding coordinate representation is holomorphic. We denote the set of holomorphic mappings from M to N by $\mathcal{O}(M,N)$. ■

Remark 2.1.13. For a map f to be smooth it suffices to check that

$$\varphi(U \cap f^{-1}(V)) \ni x \mapsto \psi(f(\varphi^{-1}(x))) \in \psi(V)$$

maps $\mathrm{int}(\varphi(U \cap f^{-1}(V)))$ into $\mathrm{int}(\psi(V))$ and is smooth in the sense of Definition 2.1.2 for each $m \in M$ and an arbitrary pair of charts $\varphi : U \to Y^+$ and $\psi : V \to Y'^+$ around m and $f(m)$ due to Remark 2.1.6 and Lemma 2.1.10. ■

Because differentiable maps have continuous differentials by their very definition, we shall also obtain tangent maps from smooth maps on manifolds with corners.

Definition 2.1.14. If M is a manifold with corners with differential structure $(U_i, \varphi_i)_{i\in I}$, which is modelled on the locally convex space Y, then the *tangent space* in $m \in M$ is defined to be $T_mM := (Y \times I_m) / \sim$, where $I_m := \{i \in I : m \in U_i\}$ and $(x,i) \sim \left(d\left(\varphi_j \circ \varphi_i^{-1}\right)(\varphi_i(m)).x, j\right)$. The set $TM := \cup_{m\in M}\{m\} \times T_mM$ is called the *tangent bundle* of M. Note that the tangent spaces T_mM are isomorphic for all $m \in M$, including the points in ∂M. ■

Proposition 2.1.15. *The tangent bundle TM is a manifold with corners and the map $\pi : TM \to M$, $(m,[x,i]) \mapsto m$ is smooth.*

Proof. Fix a differential structure $(U_i, \varphi_i)_{i\in I}$ on M. Then each U_i is a manifold with corners with respect to the differential structure (U_i, φ_i) on U_i. We endow each TU_i with the topology induced from the mappings

$$\begin{aligned}\mathrm{pr}_1 &: TU_i \to M, \quad (m,v) \mapsto m\\ \mathrm{pr}_2 &: TU_i \to Y, \quad (m,v) \mapsto v,\end{aligned}$$

and endow TM with the topology making each $TU_i \hookrightarrow TM$, $(m,v) \mapsto (x,[v,i])$ a topological embedding. Then $\varphi_i \circ \mathrm{pr}_1 \times \mathrm{pr}_2 : TU_i \to \varphi(U_i) \times Y$ defines a differential structure on TM and from the definition it follows immediately that π is smooth. ■

Corollary 2.1.16. *If M and N are manifolds with corners, then a map $f : M \to N$ is C^1 if $f(\mathrm{int}(M)) \subseteq \mathrm{int}(N)$, $f_{\mathrm{int}} := f|_{\mathrm{int}(M)}$ is C^1 and $Tf_{\mathrm{int}} : T(\mathrm{int}(M)) \to T(\mathrm{int}(N)) \subseteq TN$ extends continuously to TM. If, in addition, f is C^n for $n \geq 2$, then the* tangent map

$$Tf : TM \to TN, \quad (m,[x,i]) \mapsto \left(f(m), \left[d\left(\varphi_j \circ f \circ \varphi_i^{-1}\right)(\varphi_i(m)).x, j\right]\right)$$

is well-defined and C^{n-1}. ■

Definition 2.1.17. If M is a manifold with corners, then for $n \in \mathbb{N}_0$ the *higher tangent bundles* T^nM are the inductively defined manifolds with corners $T^0M := M$ and $T^nM := T\left(T^{n-1}M\right)$. If N is a manifold with corners and $f : M \to N$ is C^n, then the *higher tangent maps* $T^mf : T^mM \to T^mN$ are the maps defined inductively by $T^0f := f$ and $T^mf := T(T^{m-1}f)$ if $1 < m$. ■

Corollary 2.1.18. *If M, N and L are manifolds with corners and $f : M \to N$ and $g : N \to L$ with $f(\mathrm{int}(M)) \subseteq \mathrm{int}(N)$ and $g(\mathrm{int}(N)) \subseteq \mathrm{int}(L)$ are C^n, then $f \circ g : M \to L$ is C^n and we have $T^m(g \circ f) = T^mf \circ T^mg$ for all $m \leq n$.* ■

Definition 2.1.19. If M is a manifold with corners and TM is its tangent bundle, then a *vector field* on M is a smooth mapping $X : M \to TM$ such that $X(m) \in T_mM$. We denote the space of all vector fields on M by $\mathcal{V}(M)$. It is a vector space with respect to $(X+Y)(m) = X(m) + Y(m)$ and $(\lambda \cdot X)(m) = \lambda \cdot X(m)$. ■

We finally observe that we also have smooth partitions of unity for finite-dimensional manifolds with corners. This will be a very useful tool in many constructions.

Proposition 2.1.20. *If M is a finite-dimensional paracompact manifold with corners and $(U_i)_{i\in I}$ is a locally finite open cover of M, then there exists a smooth partition of unity $(f_i)_{i\in I}$ subordinated to this open cover.*

Proof. The construction in [Hi76, Theorem 2.1] actually yields smooth functions $f_i : U_i \to \mathbb{R}$ also in the sense of Definition 2.1.12. ■

2.2 Spaces of mappings

This section provides the background for the topological treatments of mapping spaces in the following chapters. The general philosophy in these chapters is to use the existing results for mapping spaces whenever possible and reduce the occurring questions of continuity (and differentiability, which we view as a special case of continuity) to mapping spaces.

The topology underlying all definitions will always be the compact-open topology. This topology on spaces of continuous mappings leads also to topologies on spaces of smooth mappings and of differential forms, which we shall introduce now.

Definition 2.2.1. If X is a Hausdorff space and Y is a topological spaces, then the *compact-open topology* on the space of continuous functions is defined as the topology generated by the sets of the form

$$\lfloor C,W\rfloor := \{f\in C(X,Y) : f(C)\subseteq W\},$$

where C runs over all compact subsets of X and W runs over all open subsets of Y. We write $C(X,Y)_c$ for the space $C(X,Y)$ endowed with the compact-open topology.

If G is a topological group, then $C(X,G)$ is a group with respect to pointwise group operation. Furthermore, the topology of compact convergence coincides with the compact-open topology [Bo89a, Theorem X.3.4.2] and thus $C(X,G)_c$ is again a topological group. A basis of unit neighbourhoods of this topology is given by $\lfloor C,W\rfloor$, where C runs over all compact subsets of X and W runs over all open unit neighbourhoods of G. If X itself is compact, then this basis is already given by $\lfloor X,W\rfloor$, where W runs over all unit neighbourhoods of G.

If Y is a locally convex space, then $C(X,Y)$ is a vector space with respect to pointwise operations. The preceding discussion implies that addition is continuous and scalar multiplication is also continuous. Since its topology is induced by the seminorms

$$p_C : C(X,Y)\to\mathbb{K},\quad f\mapsto \sup\nolimits_{x\in C}\{p(f(x))\},$$

where C runs over all compact subsets of X and p runs over all seminorms, defining the topology on Y, we see that $C(X,Y)_c$ is again locally convex.

If M and N are manifolds with corners, then every smooth map $f : M \to N$ defines a sequence of continuous map $T^n f : T^n M \to T^n N$ on the iterated tangent bundles. We thus obtain an inclusion

$$C^\infty(M,N) \hookrightarrow \prod_{n=0}^{\infty} C(T^n M, T^n M)_c, \quad f \mapsto (T^n f)_{n\in\mathbb{N}}$$

and we define the C^∞*-topology* on $C^\infty(M,N)$ to be the initial topology induced from this inclusion. For a locally convex space Y we thus get a locally convex vector topology on $C^\infty(M,Y)$.

If $\mathcal{E} = (Y, \xi : E \to X)$ is a continuous vector bundle and $S_c(\mathcal{E})$ is the set of continuous sections, then we have an inclusion $S_c(\mathcal{E}) \hookrightarrow C(X,E)$ and we thus obtain a topology on $S_c(\mathcal{E})$. If $\mathcal{E}$ is also smooth, then we have an inclusion $S(\mathcal{E}) \hookrightarrow C^\infty(M,E)$, inducing a topology $S(\mathcal{E})$, which we also call C^∞-topology. ■

Remark 2.2.2. If M is a manifold with corners and Y is a locally convex space, then we can describe the C^∞-topology on $C^\infty(M,Y)$ alternatively as the initial topology with respect to the inclusion

$$C^\infty(M,Y) \hookrightarrow \prod_{n=0}^{\infty} C(T^n M, Y), \quad f \mapsto (d^n f)_{n\in\mathbb{N}},$$

where $d^n f = \mathrm{pr}_{2^n} \circ T^n f$. In fact, we have $Tf = (f, df)$ and we can inductively write $T^n f$ in terms of $d^l f$ for $l \leq n$. This implies for a map into $C^\infty(M,Y)$ that its composition with each d^n is continuous if and only if its composition with all T^n is continuous. Because the initial topology is characterised by this property, the topologies coincide. ■

Definition 2.2.3. If $\mathcal{E} = (Y, \xi : E \to M)$ is a smooth vector bundle and $p \in \mathbb{N}_0$, then a $\mathcal{E}$*-valued* p*-form* on M is a function ω which associates to each $m \in M$ a p-linear alternating map $\omega_m : (T_m M)^p \to E_m$ such that in local coordinates the map

$$(m, X_{1,m}, \ldots, X_{p,m}) \mapsto \omega_m(X_{1,m}, \ldots, X_{p,m})$$

is smooth. We denote by

$$\Omega^p(M,\mathcal{E}) := \{\omega : \bigcup_{m\in M} (T_m M)^p \to E : \omega \text{ is a } \mathcal{E} \text{ valued } p\text{-form on } M\}$$

the space of $\mathcal{E}$-valued p-forms on M which has a canonical vector space structure induced from pointwise operations. ■

Remark 2.2.4. If $\mathcal{E} = (Y, \xi : E \to M)$ is a smooth vector bundle over the finite-dimensional manifold M, then each $\mathcal{E}$-valued p-form ω maps vector fields $X_1, \ldots, X_p$ to a smooth section $\omega.(X_1, \ldots, X_p) := \omega \circ (X_1 \times \cdots \times X_p)$ in $S(\mathcal{E})$, which is $C^\infty(M, \mathbb{R})$-linear by definition. Conversely, any alternating $C^\infty(M)$-linear map $\omega : \mathcal{V}^p(M) \to S(\mathcal{E})$ determines uniquely an element of $\Omega^p(M, \mathcal{E})$ by setting

$$\omega_m(X_{1,m}, \ldots, X_{n,m}) := \omega(\widetilde{X}_1, \ldots, \widetilde{X}_p)(m),$$

where $\widetilde{X_i}$ is an extension of $X_{i,m}$ to a smooth vector field. That $\omega_m(X_{1,m}, \ldots, X_{p,m})$ does not depend on the choice of this extension follows from the $C^\infty(M, \mathbb{R})$-linearity of ω, if one expands different choices in terms of basis vector fields. Note that the assumption on M to be finite-dimensional is crucial for this argument. ■

Remark 2.2.5. If $\mathcal{E}$ is a smooth vector bundle, then a 0-form is in particular a smooth section, whence a smooth map on M, and a 1-from defines in particular a smooth mapping on TM. We thus have canonical injections

$$\Omega^0(M, \mathcal{E}) \hookrightarrow C^\infty(M, E)$$
$$\Omega^1(M, \mathcal{E}) \hookrightarrow C^\infty(TM, E)$$

and we use this to endow $\Omega^0(M, \mathcal{E})$ and $\Omega^1(M, \mathcal{E})$ with a locally convex vector topology. Furthermore, since the conditions on ω in the previous definition are closed, these embeddings are closed. ■

We now consider the continuity properties of some very basic maps, i.e., restriction maps and gluing maps. These maps we shall encounter often in the sequel.

Lemma 2.2.6. *If $\mathcal{E}$ is a smooth vector bundle over M and $U \subseteq M$ is open and $\mathcal{E}_U = \mathcal{E}|_U$ is the restricted vector bundle, then the restriction map* $\mathrm{res}_U : S(\mathcal{E}) \to S(\mathcal{E}_U)$, $\sigma \mapsto \sigma|_U$ *is continuous. If, moreover, $\overline{U}$ is a manifold with corners, then the restriction map* $\mathrm{res}_{\overline{U}} : S(\mathcal{E}) \to S(\mathcal{E}_{\overline{U}})$, $\sigma \mapsto \sigma|_{\overline{U}}$ *is continuous.*

Proof. Because each compact $C \subseteq T^n U$ or $C' \subseteq T^n \overline{U}$ is also compact in $T^n M$, this follows directly from the definition of the C^∞-topology. ■

Proposition 2.2.7. *If $\mathcal{E}$ is a smooth vector bundle over the finite-dimensional manifold with corners M and $S(\mathcal{E})$ is the vector space of smooth sections with pointwise operations, then the C^∞-topology is a locally convex vector topology on $S(\mathcal{E})$. Furthermore, if $(U_i)_{i \in I}$ is an open cover of M such that each $\overline{U}_i$ is a manifold with corners and $\mathcal{E}_i := \mathcal{E}|_{\overline{U}_i}$ denotes the restricted bundle, then the C^∞-topology on $S(\mathcal{E})$ is initial with respect to*

$$\mathrm{res} : S(\mathcal{E}) \to \prod_{i \in I} S(\mathcal{E}_i), \quad \sigma \mapsto (\sigma|_{\overline{U}_i})_{i \in I}. \tag{2.1}$$

Proof. By choosing an open cover $(U_i)_{i\in I}$ of M such that each $\overline{U}_i$ is a trivialising manifold with corners, the second assertion implies the first, because then $S(\mathcal{E}_i) \cong C^\infty(\overline{U}_i, Y)$. Since $T^n\overline{U}_i \hookrightarrow T^nM$ is a closed embedding it is proper and thus for each compact $C \subseteq T^nM$, $C \cap T^n\overline{U}_i$ is also compact. Hence, if

$$\lfloor C_1, W_1 \rfloor \cap \cdots \cap \lfloor C_l, W_l \rfloor$$

is a basic open subset in $C(T^nM, T^nE)_c$, then

$$\lfloor C_1 \cap T^nU_i, W_1 \rfloor \cap \cdots \cap \lfloor C_l \cap T_nU_i, W_l \rfloor$$

is an open basic neighbourhood in $C(T^nU_i, T^nE)_c$ for each $i \in I$. Now it follows directly from the definition of the C^∞-topology on $S(\mathcal{E})$ that it is initial. ■

Corollary 2.2.8. *The restriction maps* res_U *and* $\mathrm{res}_{\overline{U}}$ *from Lemma 2.2.6 are smooth.* ■

Proposition 2.2.9. *If $\mathcal{E}$ is a smooth vector bundle over the finite-dimensional manifold with corners M, $\mathfrak{U} = (U_i)_{i\in I}$ is an open cover of M such that each $\overline{U}_i$ is a manifold with corners and $\mathcal{E}_i := \mathcal{E}|_{\overline{U}_i}$ denotes the restricted bundle, then*

$$S_{\overline{\mathfrak{U}}}(\mathcal{E}) = \{(\sigma_i)_{i\in I} \in \bigoplus_{i\in I} S(\mathcal{E}_i) : \sigma_i(x) = \sigma_j(x) \text{ for all } x \in \overline{U}_i \cap \overline{U}_j\}$$

is a closed subspace of $\bigoplus_{i\in I} S(\mathcal{E}_i)$ *and the* gluing map

$$\mathrm{glue} : S_{\overline{\mathfrak{U}}}(\mathcal{E}) \to S(\mathcal{E}), \quad \mathrm{glue}((\sigma_i)_{i\in I})(x) = \sigma_i(x) \textit{ if } x \in \overline{U}_i \tag{2.2}$$

is inverse to the restriction map (2.1).

Proof. Since evaluation maps are continuous in the C^∞-topology and $S_{\overline{\mathfrak{U}}}(\mathcal{E})$ can be written as an intersection of kernels of evaluation maps, it is closed. Furthermore, it is immediate that glue is a linear inverse to the restriction map. That the restriction map is open follows again from the fact that $T^n\overline{U}_i \subseteq T^nM$ is closed an thus glue is continuous. ■

Corollary 2.2.10. *If $\mathcal{E}$ is a smooth vector bundle over the finite-dimensional manifold with corners M, $\mathfrak{U} = (U_i)_{i\in I}$ is an open cover of M and $\mathcal{E}_i := \mathcal{E}|_{U_i}$ denotes the restricted bundle, then*

$$S_{\mathfrak{U}}(\mathcal{E}) = \{(\sigma_i)_{i\in I} \in \bigoplus_{i\in I} S(\mathcal{E}_i) : \sigma_i(x) = \sigma_j(x) \text{ for all } x \in U_i \cap U_j\}$$

is a closed subspace of $\bigoplus_{i\in I} S(\mathcal{E}_i)$ *and the* gluing map

$$\mathrm{glue} : S_{\mathfrak{U}}(\mathcal{E}) \to S(\mathcal{E}), \quad \mathrm{glue}((\sigma_i)_{i\in I})(x) = \sigma_i(x) \textit{ if } x \in U_i \tag{2.3}$$

is inverse to the restriction map.

Proof. Again, $S_{\mathfrak{U}}(\mathcal{E})$ can be written as the intersection of kernels and glue is clearly linear and bijective. Furthermore, choose an open cover $(V_j)_{j\in J}$ such that each $\overline{V}_j$ is a manifold with corners and $\overline{V}_j \subseteq U_{i(j)}$ for some $i(j) \in I$ and let $\mathcal{E}_j := \mathcal{E}|_{\overline{V}_j}$ be the restricted bundle. Then $S(\mathcal{E}_i) \to S(\mathcal{E}_j)$, $\sigma \mapsto \sigma|_{\overline{V}_j}$ is continuous and

$$\operatorname{glue}((\sigma_i)_{i\in I}) = \operatorname{glue}(\,(\sigma_{i(j)}\big|_{\overline{V}_j})_{j\in J})$$

shows that glue is continuous. ■

After having introduced a locally convex vector topology on $C^\infty(M,Y)$ for Y a locally convex space in Definition 2.2.1, we now wish to have that $C^\infty(M,K)$ is a Lie group if K is so. This will not hold in general, we have to restrict to compact M for this purpose. This will be the main reason for working with bundles over compact base spaces in the following chapters.

In order to show that $C^\infty(M,K)$ is a Lie group we follow the way from [Gl02b] and [Ne01].

Lemma 2.2.11. *If M is a finite-dimensional manifold with corners and X and Y are locally convex spaces, then there is an isomorphism $C^\infty(M,X\times Y)\cong C^\infty(M,X)\times C^\infty(M,Y)$.*

Proof. The proof of [Gl02b, Lemma 3.4] carries over without changes. ■

Lemma 2.2.12. *If M and N are finite-dimensional manifolds with corners, Y is locally convex and $f: N\to M$ is smooth, then the map $C^\infty(M,Y)\to C^\infty(N,Y)$, $\gamma\mapsto\gamma\circ f$ is continuous.*

Proof. The proof of [Gl02b, Lemma 3.7] carries over without changes. ■

Lemma 2.2.13. *If M is a finite-dimensional manifold with corners and Y is a locally convex space, then the map $C^\infty(M,Y)\to C^\infty(T^nM,T^nY)$, $\gamma\mapsto T^n\gamma$ is continuous.*

Proof. The proof of [Gl02b, Lemma 3.8] carries over for $n=1$, where [Gl02b, Lemma 3.7] has to be substituted by Lemma 2.2.12 and [Gl02b, Lemma 3.4] has to be substituted by Lemma 2.2.11. The assertion follows from an easy induction. ■

Lemma 2.2.14. *If X is a Hausdorff space, Y and Z are locally convex spaces, $U\subseteq Y$ is open and $f: X\times U\to Z$ is continuous, then the map*

$$f_\sharp : C(X,U)_c \to C(X,Z)_c,\ \ \gamma\mapsto f\circ(\mathrm{id}_X,\gamma)$$

is continuous.

Proof. Since the topology of compact convergence and the compact-open topology coincide on $C(X,X)$ and $C(X,Y)$ [Bo89a, Theorem X.3.4.2], this is [Gl02b, Lemma 3.9]. ■

Lemma 2.2.15. *If M is a finite-dimensional manifold with corners, X and Y are locally convex spaces, $U \subseteq X$ is open and $f : M \times U \to Y$ is smooth, then the mapping*

$$f_\sharp : C^\infty(M, U) \to C^\infty(M, Y), \quad \gamma \mapsto f \circ (\mathrm{id}_M, \gamma)$$

is continuous.

Proof. For $\gamma \in C^\infty(M, U)$ we have

$$T(f_\sharp \gamma) = T(f \circ (\mathrm{id}_M, \gamma)) = Tf \circ T(\mathrm{id}_M, \gamma) = Tf \circ (\mathrm{id}_{TM}, T\gamma) = (Tf)_\sharp (T\gamma)$$

and thus inductively

$$\begin{aligned} T^n(f_\sharp \gamma) &= T\big(T^{n-1}(f_\sharp \gamma)\big) = T\big((T^{n-1}f)_\sharp T^{n-1}\gamma\big) \\ &= T\big(T^{n-1}f \circ (\mathrm{id}_{T^{n-1}M}, T^{n-1}\gamma)\big) = T^n f \circ (\mathrm{id}_{T^n M}, T^n \gamma) = \big(T^n f\big)_\sharp T^n \gamma. \end{aligned}$$

Now, we can write the map $\gamma \mapsto T^n(f_\sharp \gamma)$ as the composition of the two maps $\gamma \mapsto (\mathrm{id}_{T^n M}, T^n \gamma)$ and $(\mathrm{id}_{T^n M}, T^n \gamma) \mapsto (T^n f)_\sharp T^n \gamma$ which are continuous by Lemma 2.2.13 and Lemma 2.2.14. Hence, $f_\sharp$ is continuous, because a map from any topological space to $C^\infty(M, Y)$ is continuous if all compositions with $d^n = \mathrm{pr}_{2^n} \circ T^n$ are continuous. ∎

Proposition 2.2.16. *a) If M is a compact manifold with corners, X and Y are locally convex spaces, $U \subseteq X$ is open and $f : M \times U \to Y$ is smooth, then the mapping $f_\sharp : C^\infty(M, U) \to C^\infty(M, Y)$, $\gamma \mapsto f \circ (\mathrm{id}_M, \gamma)$ is smooth.*
b) If, in addition, X and Y are complex vector spaces and $f_m : U \to Y$, $m \mapsto f(m, x)$ is holomorphic for all $m \in M$, then $f_\sharp$ is holomorphic.

Proof. a) (cf. [Ne01, Proposition III.7]) We claim that

$$d^n(f_\sharp) = (d_2^n f)_\sharp \tag{2.4}$$

holds for all $n \in \mathbb{N}_0$, where $d_2^n f(x, y).v := d^n f(x, y).(0, v)$. This claim immediately proves the assertion due to Lemma 2.2.15.

To verify (2.4) we perform an induction on n. The case $n = 0$ is trivial, hence assume that (2.4) holds for $n \in \mathbb{N}_0$ and take

$$\gamma \in C^\infty(M, U) \times C^\infty(M, X)^{n-1} \cong C^\infty(M, U \times X^{n-1})$$

and

$$\eta \in C^\infty(M, X)^n \cong C^\infty(M, X^n).$$

Then $\mathrm{im}(\gamma) \subseteq U \times X^{n-1}$ and $\mathrm{im}(\eta) \subseteq X^n$ are compact and there exists an $\varepsilon > 0$ such that

$$\mathrm{im}(\gamma) + (-\varepsilon, \varepsilon)\mathrm{im}(\eta) \subseteq U \times X^{n-1}.$$

Hence, $\gamma + h\eta \in C^\infty(M, U \times X^{n-1})$ for all $h \in (-\varepsilon, \varepsilon)$ and we calculate

$$\begin{aligned}
\big(d(d^n f_\sharp)(\gamma,\eta)\big)(x) &= \lim_{h\to 0} \frac{1}{h}\Big(\big(d^n f_\sharp(\gamma + h\eta) - d^n f_\sharp(\gamma)\big)(x)\Big)\\
&\overset{i)}{=} \lim_{h\to 0} \frac{1}{h}\Big(d_2^n f\big(x,\gamma(x) + h\eta(x)\big) - d_2^n f\big(x,\gamma(x)\big)\Big)\\
&\overset{ii)}{=} \lim_{h\to 0} \int_0^1 d_2\bigg(\Big(d_2^n f\big(x,\gamma(x) + th\,\eta(x)\big)\Big), \eta(x)\bigg) dt\\
&\overset{iii)}{=} \int_0^1 \lim_{h\to 0} d_2\bigg(\Big(d_2^n f\big(x,\gamma(x) + th\,\eta(x)\big)\Big), \eta(x)\bigg) dt\\
&= d_2^{n+1} f\big(x,\gamma(x),\eta(x)\big) = \big(d_2^{n+1} f\big)_\sharp(\gamma,\eta)(x),
\end{aligned}$$

where $i)$ holds by the induction hypothesis, $ii)$ holds by the Fundamental Theorem of Calculus [Gl02a, Theorem 1.5] and $iii)$ holds due to the differentiability of parameter-dependent Integrals (cf. [GN07a]).

b) The formula $d(f_\sharp) = (d_2 f)_\sharp$ shows that $d(f_\sharp)$ is complex linear. ■

Corollary 2.2.17. *If M is a compact manifold with corners, X and Y are locally convex spaces, $U \subseteq X$ are open and $f : U \to Y$ is smooth (respectively, holomorphic), then the push-forward $f_* : C^\infty(M, U) \to C^\infty(M, Y)$, $\gamma \mapsto f \circ \gamma$ is a smooth (respectively, holomorphic) map.*

Proof. Define $\tilde{f} : M \times U \to Y$, $(x, v) \mapsto f(x)$ and apply Proposition 2.2.16. ■

Remark 2.2.18. If M is a complex manifold with corners and Y is a locally convex complex vector space, then $\mathcal{O}(M, Y)$ is a closed subspace of $C^\infty(M, Y)$. In fact, the requirement that $df(x)$ is complex linear is a closed condition as an equational requirement on $df(x)$ in the topology defined in Definition 2.2.1. ■

We now see that $C^\infty(M, K)$ is in fact a Lie group, provided that M is compact. Along with this assertion, we also consider the case when K is a complex Lie group.

Theorem 2.2.19 (Lie group structure on $C^\infty(M,K)$). *Let M be a compact manifold with corners, K be a Lie group and let $\varphi : W \to \varphi(W) \subseteq \mathfrak{k}$ be a convex centred chart of K. Furthermore denote $\varphi_* : C^\infty(M, W) \to C^\infty(M, \mathfrak{k})$, $\gamma \mapsto \varphi \circ \gamma$.*

a) If M and K are smooth, then φ_* induces a locally convex manifold structure on $C^\infty(M, K)$, turning it into a Lie group with respect to pointwise operations.

b) If M is smooth and K is complex, then φ_* induces a complex manifold structure on $C^\infty(M, K)$, turning it into a complex Lie group with respect to pointwise operations.

c) *If M and K are complex, then the restriction of φ_* to $\mathcal{O}(M,W)$ induces a complex manifold structure on $\mathcal{O}(M,K)$, turning it into a complex Lie group with respect to pointwise operations, modelled on $\mathcal{O}(M,\mathfrak{k})$.*

Proof. Using Corollary 2.2.17 and Proposition 2.2.16, the proof of the smooth case in [Gl02b, 3.2] yields a). Since Proposition 2.2.16 also implies that the group operations are holomorphic, b) is now immediate. Using the same argument as in a), we deduce c), since φ_* maps $\mathcal{O}(M,W)$ bijectively to $\mathcal{O}(M,\varphi(W))$, which is open in $\mathcal{O}(M,\mathfrak{k})$. ■

We now derive the smoothness of the restriction and gluing maps for Lie group valued functions (cf. Lemma 2.2.6 and Proposition 2.2.9). This will be important tools in many following constructions.

Lemma 2.2.20. *If M is a compact manifold with corners, K is a Lie group and $\overline{U}\subseteq M$ is a manifold with corners, then the restriction*

$$\mathrm{res}: C^\infty(M,K)\to C^\infty(\overline{U},K),\quad \gamma\mapsto\gamma|_{\overline{U}}$$

is a smooth homomorphism of Lie groups.

Proof. If $\varphi: W\to\varphi(W)\subseteq\mathfrak{k}$ is a convex centred chart, then the coordinate representation on $C^\infty(M,W)$ is given by $C^\infty(M,\varphi(W))\to C^\infty(\overline{U},\varphi(W))$, $\eta\mapsto\eta|_{\overline{U}}$, which is smooth. ■

Proposition 2.2.21. *Let K be a Lie group, M be a compact manifold with corners with an open cover $\mathfrak{V}=(V_1,\ldots,V_n)$ such that $\overline{\mathfrak{V}}=(\overline{V_1},\ldots,\overline{V_n})$ is a cover by manifolds with corners. Then*

$$G_{\overline{\mathfrak{V}}}:=\{(\gamma_1,\ldots,\gamma_n)\in\prod_{i=1}^n C^\infty(\overline{V}_i,K):\gamma_i(x)=\gamma_j(x)\ \text{ for all }\ x\in\overline{V}_i\cap\overline{V}_j\}$$

is a closed subgroup of $\prod_{i=1}^n C^\infty(\overline{V}_i,K)$, which is a Lie group modelled on the closed subspace

$$\mathfrak{g}_{\mathfrak{V}}:=\{(\eta_1,\ldots,\eta_n)\in\prod_{i=1}^n C^\infty(\overline{V}_i,\mathfrak{k}):\eta_i(x)=\eta_j(x)\ \text{ for all }\ x\in\overline{V}_i\cap\overline{V}_j\}$$

of $\bigoplus_{i=1}^n C^\infty(\overline{V}_i,\mathfrak{k})$ and the gluing map

$$\mathrm{glue}: G_{\overline{\mathfrak{V}}}\to C^\infty(M,K),\quad \mathrm{glue}(\gamma_1,\ldots,\gamma_n)=\gamma_i(x)\ \textit{if}\ x\in\overline{V}_i$$

is an isomorphism of Lie groups.

Proof. Since the evaluation map is continuous, $G_{\overline{\mathfrak{V}}}$ is closed as it can be written as an intersection of closed subgroups. Let $\varphi : W \to \varphi(W) \subseteq \mathfrak{k}$ be a convex centred chart of K. Then

$$O = \{(\gamma_1, \ldots, \gamma_n) \in C^\infty(\overline{V}_i, K) : \gamma_i(\overline{V}_i) \subseteq W\}$$

is an open unit neighbourhood in $\prod_{i=1}^n C^\infty(\overline{V}_i, K)$ and

$$O' = \{(\gamma_1, \ldots, \gamma_n) \in C^\infty(\overline{V}_i, \mathfrak{k}) : \gamma_i(\overline{V}_i) \subseteq \varphi(W)\}$$

is an open zero neighbourhood in $\mathfrak{g}_{\overline{\mathfrak{V}}}$ and the chart $(\gamma_1, \ldots, \gamma_n) \mapsto (\varphi \circ \gamma_1, \ldots, \varphi \circ \gamma_n)$ defines a Lie group structure on $G_{\overline{\mathfrak{V}}}$ as in Theorem 2.2.19.

Clearly, glue is an isomorphism of abstract groups and because the restriction map, provided by Lemma 2.2.20, is smooth, it suffices to show that glue is smooth on a unit neighbourhood. Since the charts are given by push-forwards, the coordinate representation of glue on $O \cap G_{\overline{\mathfrak{V}}}$ is given by the gluing map on the Lie algebra, which is smooth (cf. Proposition 2.2.9). ■

We finally collect some facts on actions on spaces of smooth mappings arising as pull-backs and push-forwards of smooth mappings. These facts we will frequently refer to in the sequel.

Proposition 2.2.22. *Let X, Y, Z be locally convex spaces, $U \subseteq Z$ be an open subset, M be a locally convex manifold without boundary and $f : U \times M \times X \to Y$ be smooth. Then the push forward*

$$f_* : U \times C^\infty(M, X) \to C^\infty(M, Y), \quad f_*(z, \xi)(m) = f(z, m, \xi(m))$$

is smooth.

Proof. This is a special case of [Gl04, Proposition 4.16]. ■

Corollary 2.2.23. *If G is a Lie group that acts smoothly on some locally convex space Y and M is a compact manifold without boundary, then the induced pointwise action*

$$C^\infty(M, G) \times C^\infty(M, Y) \to C^\infty(M, Y), \quad (\gamma.\xi)(m) = \gamma(m).\xi(m)$$

is smooth.

Proof. Taking $f_1 : M \times Y \to Y$, $(m, y) \mapsto \gamma(m).y$ for a fixed $\gamma \in C^\infty(M, G)$, Proposition 2.2.22 shows that $C^\infty(M, G)$ acts by continuous linear automorphisms. If we identify some unit neighbourhood $U \subseteq C^\infty(M, G)$ with an open subset of its modelling space, then Proposition 2.2.22, applied to $f_2 : U \times M \times Y \to Y$, $(\gamma, m, x) \mapsto \gamma(m).x$, yields the assertion, because it suffices for an action to be smooth on some unit neighbourhood by Lemma A.3.3. ■

Lemma 2.2.24. *If M and N are smooth locally convex manifolds without boundary, Y is a locally convex space and $f \in C^\infty(N, M)$ is smooth, then the pull-back*

$$f^* : C^\infty(M, Y) \to C^\infty(N, Y), \quad \gamma \mapsto \gamma \circ f$$

is linear and continuous.

Proof. It is immediate that f^* is linear and by [Gl04, Lemma 4.11], it is continuous. ■

Lemma 2.2.25. *If G is a Lie group, M is a finite-dimensional manifold without boundary with a smooth action $G \times M \to M$ and and Y is a locally convex space, then the pull-back action*

$$G \times C^\infty(M, Y) \to C^\infty(M, Y), \quad (g.\eta)(m) = \eta(g^{-1}.m)$$

is smooth. In particular, if M is compact, then the action

$$\mathrm{Diff}(M) \times C^\infty(M, Y) \to C^\infty(M, Y), \quad g.\eta = \eta \circ g^{-1}$$

is smooth.

Proof. Considering the trivial vector bundle $\mathcal{E}_Y = (Y, \mathrm{pr}_1 : M \times Y \to M)$ with the trivial G-action on M, this is a special case of [Gl06, Proposition 6.4]. ■

Lemma 2.2.26. *If M is a smooth compact manifold without boundary and Y is a locally convex space, then the action*

$$\mathrm{Diff}(M) \times \Omega^1(M, Y) \to \Omega^1(M, Y), \quad g.\omega = (g^{-1})^*\omega = \omega \circ Tg^{-1}$$

is smooth.

Proof. This follows from [Gl06, Corollary 6.6]. ■

Proposition 2.2.27. *If M is a compact manifold without boundary and K is a Lie group, then the action*

$$\mathrm{Diff}(M) \times C^\infty(M, K) \to C^\infty(M, K), \quad g.\gamma = \gamma \circ g^{-1} \tag{2.5}$$

is smooth.

Proof. This is [Gl06, Proposition 10.3] ■

2.3 Extensions of smooth maps

This section draws on a suggestion by Helge Glöckner and was inspired by [Br92, Chapter IV]. We relate the notions of differentiability on sets with dense interior, introduced in Definition 2.1.2, to the usual notion of differentiability on a non-open subset $U \subseteq \mathbb{R}^n$ (cf. Remark 2.1.8).

We will see that, at least under some mild requirements, this notion coincides with the definition given in Definition 2.1.2.

We shall use the following observation, also known as *exponential law* or *Cartesian closedness principle* to reduce the extension of smooth maps from $[0,1]^n$ to $\mathbb{R}^n$ to the extension of smooth maps from $[0,1]$ to $\mathbb{R}$.

Proposition 2.3.1. *If X, Y are Fréchet spaces, $U_1 \subseteq X$ and $U_2 \subseteq \mathbb{R}^n$ have dense interior, then we have a linear isomorphism*

$$^\wedge : C^\infty(U_1 \times U_2, Y) \to C^\infty(U_1, C^\infty(U_2, Y)), \quad f^\wedge(x)(y) = f(x,y).$$

Proof. First we check that $f^\wedge$ actually is an element of $C^\infty(U_1, C^\infty(U_2, Y))$. Since for open domains in Fréchet spaces, the notion of differentiability from Definition 2.1.2 and the one used in the convenient calculus coincide (cf. Remark A.1.2), [KM97, Lemma 3.12] implies that $f^\wedge(x)|_{\mathrm{int}(U_2)} \in C^\infty(\mathrm{int}(U_2), Y)$ if $x \in \mathrm{int}(U_1)$. Since $d^n f$ extends continuously to the boundary, so does $d^n(f^\wedge(x))$. So $f^\wedge|_{\mathrm{int}(U_1)}$ defines a map to $C^\infty(U_2, Y)$ which is continuous since $C(U \times V, W) \cong C(U, C(V, W))$ if V is locally compact ([Bo89a, Corollary X.3.4.2]). Next we show that we can extend it to a continuous map on U_1. If $x \in \partial U_1 \cap U_1$, then there exists a sequence $(x_i)_{i \in \mathbb{N}}$ in $\mathrm{int}(U_1)$ with $x_i \to x$ and thus $(d^n(f^\wedge(x_i)))_{i \in \mathbb{N}}$ is a Cauchy sequence in $C(T^n U_2, Y)$ since $d^n f$ is continuous. Since $C^\infty(U_2, Y)$ is complete, $(f^\wedge(x_i))_{i \in \mathbb{N}}$ converges to some $f^\wedge(x) \in C^\infty(U_2, Y)$, and this extends $f^\wedge|_{\mathrm{int}(U_1)}$ continuously. Since the inclusion $C^\infty(U_2, Y) \hookrightarrow C(U_2, Y)$ is continuous and continuous extensions are unique we know that this extension is actually given by $f^\wedge$. With Remark 2.1.5, the smoothness of $f^\wedge$ follows in the same way as the continuity. It is immediate that $^\wedge$ is linear and injective, and surjectivity follows directly from $C(X \times Y, Z) \cong C(X, C(Y, Z))$. ■

To use the previous fact we need to know that the spaces under consideration are Fréchet spaces.

Remark 2.3.2. Let M be a σ-compact finite-dimensional manifold with corners and Y be a Fréchet space. Then $C(M, Y)$ and $C^\infty(\mathrm{int}(M), Y)$ are Fréchet spaces too (cf. [GN07a]). Thus, the locally convex vector topology on $C^\infty(M, Y)$ from Definition 2.2.1 is complete, turning it into a Fréchet space. Note that this is *not* immediate if one uses the notion of smoothness on M from [Le03] or [La99] as in Remark 2.1.8. ■

We now show how smooth mappings on $[0,1]$ can be extended to $\mathbb{R}$. As said before, this will be the generic case which we will reduce the general extension problem to.

Lemma 2.3.3. *If Y is a locally convex space and $(f_n)_{n\in\mathbb{N}_0}$ is a sequence in $C^1(\mathbb{R},Y)$ such that $(f(x))_n$ converges for some $x\in\mathbb{R}$ and that $(f_n')_{n\in\mathbb{N}_0}$ converges uniformly on compact subsets to some $\bar{f}\in C(\mathbb{R},Y)$, then (f_n) converges to some $f\in C^1(\mathbb{R},Y)$ with $f'=\bar{f}$.*

Proof. This can be proved as in the case $Y=\mathbb{R}$ (cf. [Br92, Proposition IV.1.7]).■

Lemma 2.3.4. *Let Y be a Fréchet space. If $(v_n)_{n\in\mathbb{N}_0}$ is an arbitrary sequence in Y, then there exists an $f\in C^\infty(\mathbb{R},Y)$ such that $f^{(n)}(0)=v_n$ for all $n\in\mathbb{N}_0$.*

Proof. (cf. [Br92, Proposition IV.4.5] for the case $Y=\mathbb{R}$). Let $\zeta\in C^\infty(\mathbb{R},\mathbb{R})$ be such that $\operatorname{supp}(\zeta)\subseteq[-1,1]$ and $\zeta(x)=1$ if $-\frac{1}{2}\leq x\leq\frac{1}{2}$ and put $\xi(x):=x\,\zeta(x)$. Then $\operatorname{supp}(\xi)\subseteq[-1,1]$ and $\xi|_{[-\frac{1}{2},\frac{1}{2}]}=\operatorname{id}_{[-\frac{1}{2},\frac{1}{2}]}$. Since ξ^k is compactly supported, there exists for each $n\in\mathbb{N}$ an element $M_{n,k}\in\mathbb{R}$ such that $|\left(\xi^k\right)^{(n)}(x)|\leq M_{n,k}$ for all $x\in\mathbb{R}$. Now let $(p_m)_{m\in\mathbb{N}}$ be a sequence of seminorms defining the topology on Y with $p_1\leq p_2\leq\ldots$. We now choose $c_k>1$ such that $p_k(v_k)c_k^{n-k}M_{n,k}<2^{-k}$ if $n<k$. Note that this is possible since there are only finitely many inequalities for each k. Set $f_m:=\sum_{k=0}^m v_k\left(c_k^{-1}\xi(c_k\,\cdot)\right)^k$, and note that $f_0(0)=v_0$ and $f_m(0)=0$ if $m\geq 1$, which shows in particular that $(f_m(0))$ converges. We show that $f:=\lim_{m\to\infty}f_m$ has the desired properties. If $\varepsilon>0$ and $\ell\in\mathbb{N}$, we let $m_{\varepsilon,\ell}>\ell$ be such that $2^{-m_{\varepsilon,\ell}}<\varepsilon$. Thus

$$p_\ell(f_m^{(n)}-f_{m_{\varepsilon,\ell}}^{(n)})=p_\ell\Big(\sum_{k=1+m_{\varepsilon,\ell}}^{m}v_kc_k^{-k}(\xi(c_k\,\cdot\,)^k)^{(n)}\Big)$$
$$\leq\sum_{k=1+m_{\varepsilon,\ell}}^{m}p_k(v_k)c_k^{n-k}M_{n,k}\leq 2^{-m_{\varepsilon,\ell}}<\varepsilon$$

for all $m>m_{\varepsilon,\ell}$ and $n<\ell$. It follows for $n<\ell$ that the sequence $(f_m^{(n)})_{m\in\mathbb{N}}$ converges uniformly to some $f^n\in C^\infty(\mathbb{R},Y)$ and the preceding lemma implies $(f^{n-1})'=f^n$, whence $f^{(n)}=f^n$. Since ℓ was chosen arbitrarily, f is smooth. We may interchange differentiation and the limit by the preceding lemma and since $c_k\xi(c_k\,\cdot\,)$ equals the identity on a zero neighbourhood, we have $f^{(n)}(0)=\left(\lim_{m\to\infty}f_m^{(n)}\right)(0)=\lim_{m\to\infty}\left(f_m^{(n)}(0)\right)=v_n$. ■

Corollary 2.3.5. *If Y is a Fréchet space, then for each $f\in C^\infty([0,1],Y)$ there exists an $\bar{f}\in C^\infty(\mathbb{R},Y)$ with $\bar{f}\big|_{[0,1]}=f$.*

Proof. (cf. [KM97, Proposition 24.10]) For $n \in \mathbb{N}_0$ set $v_n := f^{(n)}(0)$ and $w_n := f^{(n)}(1)$. Then the preceding lemma yields $f_-, f_+ \in C^\infty(\mathbb{R}, Y)$ with $f_-^{(n)}(0) = v_n = f^{(n)}(0)$ and $f_+^{(n)}(0) = w_n = f^{(n)}(1)$. Then

$$\bar{f}(x) := \begin{cases} f_-(x) & \text{if } x < 0 \\ f(x) & \text{if } 0 \leq x \leq 1 \\ f_+(x-1) & \text{if } x > 1 \end{cases}$$

defines a function on $\mathbb{R}$ which has continuous differentials of arbitrary order and hence is smooth. ■

As indicated before, a combination of Proposition 2.3.1 and Corollary 2.3.5 enables us now to extend smooth mappings defined on $[0,1]^n$ to smooth mappings on $\mathbb{R}^n$.

Theorem 2.3.6 (Extension of smooth maps). *If Y is a Fréchet space and $f \in C^\infty([0,1]^n, Y)$, then there exists an $\bar{f} \in C^\infty(\mathbb{R}^n, Y)$ with $\bar{f}\big|_{[0,1]^n} = f$.*

Proof. Set $f_0 := f$. Using Proposition 2.3.1, we can view f_0 as an element

$$f_0 \in C^\infty\left([0,1], C^\infty\left([0,1]^{n-1}, Y\right)\right),$$

which we can extend to an element of $C^\infty\left(\mathbb{R}, C^\infty\left([0,1]^{n-1}, Y\right)\right)$ by Corollary 2.3.5 and Remark 2.3.2. This can again be seen as an element $f_1 \in C^\infty\left(\mathbb{R} \times [0,1]^{n-1}, Y\right)$. In the same manner, we obtain a map

$$f_2 \in C^\infty\left(\mathbb{R}^2 \times [0,1]^{n-2}, Y\right)$$

extending f_1 as well as f_0. Iterating this procedure for each argument results in a map $\bar{f} := f_n$ which extends each f_i and so it extends $f_0 = f$. ■

The case of manifolds with corners, more general than $[0,1]^n$, now follows from this case by a partition of unity argument.

Proposition 2.3.7. *If Y is a Fréchet space, M is a finite-dimensional manifold without boundary, $L \subseteq M$ has dense interior and is a manifold with corners with respect to the charts obtained from the restriction of the charts of M to L, then there exists an open subset $U \subseteq M$ with $L \subseteq U$ such that for each $f \in C^\infty(L, Y)$ there exists a $\bar{f} \in C^\infty(U, Y)$ with $\bar{f}\big|_L = f$.*

Proof. For each $m \in \partial L$ there exists a set L_m which is open in M and a chart $\varphi_m : L_m \to \mathbb{R}^n$ such that $\varphi_m(L \cap L_m) \subseteq \mathbb{R}^n_+$ and $\varphi_m(m) \in \partial\mathbb{R}^n_+$. Then there exists a cube

$$C_m := [x_1 - \varepsilon, x_1 + \varepsilon] \times \ldots \times [x_n - \varepsilon, x_n + \varepsilon] \subseteq \varphi_m(L \cap L_m),$$

where

$$x_i = \begin{cases} \varphi_m(m)_i & \text{if } \varphi_m(m)_i \neq 0 \\ \varepsilon & \text{if } \varphi_m(m)_i = 0 \end{cases}$$

(actually C_m is contained in $\mathbb{R}^n_+$ and shares the i-th "boundary-face" with $\mathbb{R}^n_+$ if $\varphi_m(m)_i = 0$). Then C_m is diffeomorphic to $[0,1]^n$. The diffeomorphism is defined by multiplication and addition and extends to a diffeomorphism of $\mathbb{R}^n$. We now set $U = \operatorname{int}(L) \cup \bigcup_{m \in \partial L \cap L} V_m,\ V_m := \operatorname{int}(\varphi_m^{-1}(C_m))$. Then this open cover has a locally finite refinement $(\operatorname{int}(L), (V_i')_{i \in I})$ with $V_i' \subseteq V_{m(i)}$ for some function $I \ni i \mapsto m(i) \in \partial L$. Now, choose a partition of unity $g, h, (h_i)_{i \in I}$ subordinated to the open cover $(U \backslash L, \operatorname{int}(L), (V_i')_{i \in I})$.

If $f \in C^\infty(L, M)$, then Theorem 2.3.6 yields a smooth extension f_m of $f \circ \varphi_m^{-1}|_{C_m}$ and thus $\bar{f}_m := f_m \circ \varphi_m|_{V_m}$ is smooth and extends f. We now set

$$\bar{f}(x) := h(x)\, f(x) + \sum_{i \in I} h_i(x)\, \bar{f}_{m(i)}(x),$$

where we extend f and f_m by zero if not defined. Since h (respectively, h_i) vanishes on a neighbourhood of each point in ∂L (respectively, $\partial V_{m(i)}$), this function is smooth and since $\bar{f}_m\big|_{V_m \cap L} = f|_{V_m \cap L}$ for all $m \in \partial L$, it also extends f. ■

Corollary 2.3.8. *If $U \subseteq (\mathbb{R}^n)^+$ is open, Y a Fréchet space and $f : U \to Y$ is smooth in the sense of Definition 2.1.2, then there exists an open subset $\widetilde{U} \subseteq \mathbb{R}^n$, with $U \subseteq \widetilde{U}$, such that for each $f \in C^\infty(U, Y)$ there exists an $\widetilde{f} \in C^\infty(\widetilde{U}, Y)$ with $\widetilde{f}\Big|_U = f$.* ■

Remark 2.3.9. Similar statements to the ones from this section, known as the Whitney Extension Theorem, can be found in [Wh34], [KM97, Theorem 22.17] and [KM97, Theorem 24.10]. The remarkable point in the proofs given here is that the used methods are quite elementary, up to the Cartesian closedness principle from [KM97], which we used in the proof of Proposition 2.3.1. ■

Chapter 3

The gauge group as an infinite-dimensional Lie group

This chapter introduces the gauge groups $\mathrm{Gau}(\mathcal{P})$ of a smooth principal K-bundle and describes various aspects of it as an infinite-dimensional Lie group.

The first section describes the topologisation of $\mathrm{Gau}(\mathcal{P})$, which is the starting point for any further considerations. In the second section, we describe how the topology introduced in the first section can be made accessible by reducing the determination of the homotopy groups $\pi_n(\mathrm{Gau}(\mathcal{P}))$ to the determination of $\pi_n(\mathrm{Gau}_c(\mathcal{P}))$, where $\mathrm{Gau}_c(\mathcal{P})$ is the continuous gauge group. Developing the techniques of Section 3.2 further, we obtain in the third section a nice result on smoothing continuous principal bundles and bundle equivalences. Although this section does not deal with $\mathrm{Gau}(\mathcal{P})$, we placed it here, because the ideas used in this section are similar to the ideas used in the second section. In the fourth and last section we describe how the topologisation of $\mathrm{Gau}(\mathcal{P})$ leads to a topologisation of the automorphism group $\mathrm{Aut}(\mathcal{P})$ of $\mathcal{P}$.

3.1 The Lie group topology on the gauge group

In this section we introduce the object of central interest, namely the gauge group $\mathrm{Gau}(\mathcal{P})$ of a smooth principal K-bundle $\mathcal{P}$ and describe how it can be topologised as an infinite-dimensional Lie group. We shall mostly identify the gauge group with the space of K-equivariant continuous mappings $C^\infty(P,K)^K$, where K acts on itself by conjugation from the right.

This identification allows us to topologise the gauge group very similar to mapping groups $C^\infty(M,K)$ for compact M. Since the compactness of M is the crucial point in the topologisation of mapping groups, we can not take this approach directly, because our structure groups K shall not be compact, even infinite-dimensional. The procedure in this section is motivated by the observation that for trivial bundles, $C^\infty(P,K)^K \cong C^\infty(M,K)$. In fact, if $\sigma : M \to P$ is a global section, then

$$C^{\infty}(P,K)^{K} \to C^{\infty}(M,K), \quad \gamma \mapsto \gamma \circ \sigma$$

is an isomorphism. If M is compact, then we can take this isomorphism to turn $C^{\infty}(P,K)^{K}$ into an infinite-dimensional Lie group, modelled on $C^{\infty}(M,\mathfrak{k})$.

In the case of a non-trivial bundle things are more subtle and we shall use this section to describe how the above idea generalises to non-trivial bundles.

Throughout this section we work with bundles over compact manifolds M, possibly with corners.

We first give the basic definitions of the objects under consideration.

Definition 3.1.1. If K is a topological group and $\mathcal{P}=(K,\pi:P\to M)$ is a continuous principal K-bundle, then we denote by

$$\mathrm{Aut}_c(\mathcal{P}) := \{f \in \mathrm{Homeo}(P) : \rho_k \circ f = f \circ \rho_k \text{ for all } k \in K\}$$

the group of continuous *bundle automorphisms* and by

$$\mathrm{Gau}_c(\mathcal{P}) := \{f \in \mathrm{Aut}_c(\mathcal{P}) : \pi \circ f = \pi\}$$

the group of continuous *vertical* bundle automorphisms or *continuous gauge group*. If, in addition, K is a Lie group, M is a manifold with corners and $\mathcal{P}$ is a smooth principal bundle, then we denote by

$$\mathrm{Aut}(\mathcal{P}) := \{f \in \mathrm{Diff}(P) : \rho_k \circ f = f \circ \rho_k \text{ for all } k \in K\}$$

the the group of *smooth bundle automorphisms* (or shortly *bundle automorphisms*). Then each $F \in \mathrm{Aut}(\mathcal{P})$ induces an element $F_M \in \mathrm{Diff}(M)$, given by $F_M(p\cdot K) := F(p)\cdot K$ if we identify M with P/K. This yields a homomorphism $Q:\mathrm{Aut}(\mathcal{P}) \to \mathrm{Diff}(M)$, $F \mapsto F_M$ and we denote by $\mathrm{Gau}(\mathcal{P})$ the kernel of Q and by $\mathrm{Diff}(M)_{\mathcal{P}}$ the image of Q. Thus

$$\mathrm{Gau}(\mathcal{P}) = \{f \in \mathrm{Aut}(\mathcal{P}) : \pi \circ f = \pi\},$$

which we call the group of (smooth) vertical bundle automorphisms or shortly the *gauge group* of $\mathcal{P}$. ■

As said in the introduction to this section, the gauge group is isomorphic to a group of equivariant mappings. This identification will be the key to the topologisation of the gauge group.

Remark 3.1.2. If $\mathcal{P}$ is a smooth principal K-bundle and if we denote by

$$C^{\infty}(P,K)^{K} := \{\gamma \in C^{\infty}(P,K) : \gamma(p\cdot k) = k^{-1}\cdot\gamma(p)\cdot k \text{ for all } p \in P, k \in K\}$$

the group of K-equivariant smooth maps from P to K, then the map

$$C^{\infty}(P,K)^{K} \ni f \mapsto \big(p \mapsto p \cdot f(p)\big) \in \operatorname{Gau}(\mathcal{P})$$

is an isomorphism of groups and we will mostly identify $\operatorname{Gau}(\mathcal{P})$ with $C^{\infty}(P,K)^{K}$ via this map. ■

The algebraic counterpart of the gauge group is the gauge algebra. This will serve as the modelling space for the gauge group later on.

Definition 3.1.3. If $\mathcal{P}$ is a smooth principal K-bundle, then the space

$$\begin{aligned}\mathfrak{gau}(\mathcal{P}) &:= C^{\infty}(P,\mathfrak{k})^{K} \\ &:= \{\eta \in C^{\infty}(P,\mathfrak{k})^{K} : \eta(p\cdot k) = \operatorname{Ad}(k^{-1}).\eta(p) \text{ for all } p \in P, k \in K\}\end{aligned}$$

is called the *gauge algebra* of $\mathcal{P}$. We endow it with the subspace topology from $C^{\infty}(P,\mathfrak{k})$ and with the pointwise Lie bracket. ■

It will be convenient to have different pictures of the gauge algebra in mind. We will use these pictures interchangeably and relate them in the following proposition.

Proposition 3.1.4. *Let $\mathcal{P} = (K, \pi : P \to M)$ be a smooth principal K-bundle over the finite-dimensional manifold with corners M. If $\overline{\mathcal{V}} := (\overline{V}_i, \sigma_i)_{i\in I}$ is a smooth closed trivialising system of $\mathcal{P}$ with transition functions $k_{ij} : \overline{V}_i \cap \overline{V}_j \to K$, then we denote*

$$\mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P}) := \left\{ (\eta_i)_{i\in I} \in \prod_{i\in I} C^{\infty}(\overline{V}_i, \mathfrak{k}) : \eta_i(m) = \operatorname{Ad}(k_{ij}(m)).\eta_j(m) \ \forall m \in \overline{V}_i \cap \overline{V}_j \right\}.$$

If $\mathcal{V}$ denotes the smooth open trivialising system underlying $\overline{\mathcal{V}}$, then we set

$$\mathfrak{g}_{\mathcal{V}}(\mathcal{P}) := \left\{ (\eta_i)_{i\in I} \in \prod_{i\in I} C^{\infty}(V_i, \mathfrak{k}) : \eta_i(m) = \operatorname{Ad}(k_{ij}(m)).\eta_j(m) \ \forall m \in V_i \cap V_j \right\},$$

and we have isomorphisms of topological vector spaces

$$\mathfrak{gau}(\mathcal{P}) = C^{\infty}(P,\mathfrak{k})^{K} \cong S(\operatorname{Ad}(\mathcal{P})) \cong \mathfrak{g}_{\mathcal{V}}(\mathcal{P}) \cong \mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P}).$$

Furthermore, each of these spaces is a locally convex Lie algebra in a natural way and the isomorphisms are isomorphisms of topological Lie algebras.

Proof. The last two isomorphisms are provided by Proposition 2.2.9 and Corollary 2.2.10, so we show $C^{\infty}(P,\mathfrak{k})^{K} \cong \mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})$.

For each $\eta \in C^{\infty}(P,\mathfrak{k})^{K}$ the element $(\eta_i)_{i\in I}$ with $\eta_i = \eta \circ \sigma_i$ defines an element of $\mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})$ and the map

$$\psi : C^{\infty}(P,\mathfrak{k})^{K} \to \mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P}), \quad \eta \mapsto (\eta_i)_{i\in I}$$

is continuous. In fact, $\sigma_i(m) = \sigma_j(m) \cdot k_{ji}(m)$ for $m \in \overline{V}_i \cap \overline{V}_j$ implies

$$\eta_i(m)=\eta(\sigma_i(m))=\eta(\sigma_j(m) \cdot k_{ji}(m))=\mathrm{Ad}(k_{ji}(m))^{-1}.\eta(\sigma_j(m))=\mathrm{Ad}(k_{ij}(m)).\eta_j(m)$$

and thus $(\eta_i)_{i\in I} \in \mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})$. Recall that if X is a topological space, then a map $f : X \to C^\infty(\overline{V}_i, \mathfrak{k})$ is continuous if and only if $x \mapsto d^n f(x)$ is continuous for each $n \in \mathbb{N}_0$ (Remark 2.2.2). This implies that ψ is continuous, because $d^n\eta_i = d^n\eta \circ T^n\sigma_i$ and pull-backs along continuous maps are continuous.

On the other hand, if $k_i : \pi^{-1}(\overline{V}_i) \to K$ is given by $p = \sigma_i(\pi(p)) \cdot k_i(p)$ and if $(\eta_i)_{i\in I} \in \mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})$, then the map

$$\eta : P \to \mathfrak{k}, \quad p \mapsto \mathrm{Ad}\left(k(p)\right)^{-1}.\eta_i\left(\pi(p)\right) \;\text{ if }\; \pi(p) \in \overline{V}_i$$

is well-defined, smooth and K-equivariant. Furthermore, $(\eta_i)_{i\in I} \mapsto \eta$ is an inverse of ψ and it thus remains to check that it is continuous, i.e., that

$$\mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P}) \ni (\eta_i)_{i\in I} \mapsto d^n\eta \in C(T^nP, \mathfrak{k})$$

is continuous for all $n \in \mathbb{N}_0$. If $C \subseteq T^nP$ is compact, then $(T^n\pi)(C) \subseteq T^nM$ is compact and hence it is covered by finitely many $T^nV_{i_1}, \ldots, T^nV_{i_m}$ and thus $\left(T^n\left(\pi^{-1}(\overline{V_i})\right)\right)_{i=i_1,\ldots,i_m}$ is a finite closed cover of $C \subseteq T^nP$. Hence it suffices to show that the map

$$\mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P}) \ni (\eta_i)_{i\in I} \mapsto T^n(\eta|_{\pi^{-1}(\overline{V}_i)}) \in C(T^n\pi^{-1}(\overline{V}_i), \mathfrak{k})$$

is continuous for $n \in \mathbb{N}_0$ and $i \in I$ and we may thus w.l.o.g. assume that $\mathcal{P}$ is trivial. In the trivial case we have $\eta = \mathrm{Ad}(k^{-1}).(\eta \circ \pi)$ if $p \mapsto (\pi(p), k(p))$ defines a global trivialisation. We shall make the case $n = 1$ explicit. The other cases can be treated similarly and since the formulae get quite long we skip them here.

Given any open zero neighbourhood in $C(TP, \mathfrak{k})$, which we may assume to be $\lfloor C, V \rfloor$ with $C \subseteq TP$ compact and $0 \in V \subseteq \mathfrak{k}$ open, we have to construct an open zero neighbourhood O in $C^\infty(M, \mathfrak{k})$ such that $\varphi(O) \subseteq \lfloor C, V \rfloor$. For $\eta' \in C^\infty(M, \mathfrak{k})$ and $X_p \in C$ we get with Lemma A.3.10

$$d(\varphi(\eta'))(X_p) = \mathrm{Ad}(k^{-1}(p)).d\eta'(T\pi(X_p)) - [\delta^l(k)(X_p), \mathrm{Ad}(k^{-1}(p)).\eta'(\pi(p))].$$

Since $\delta^l(C) \subseteq \mathfrak{k}$ is compact, there exists an open zero neighbourhood $V' \subseteq \mathfrak{k}$ such that

$$\mathrm{Ad}(k^{-1}(p)).V' + [\delta^l(k)(X_p), \mathrm{Ad}(k^{-1}(p)).V'] \subseteq V$$

for each $X_p \in C$. Since $T\pi : TP \to TM$ is continuous, $T\pi(C)$ is compact and we may set $O = \lfloor T\pi(C), V' \rfloor$.

That $\mathfrak{g}_{\mathcal{V}}(\mathcal{P})$ and $\mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})$ are locally convex Lie algebras follows because they are closed subalgebras of $\prod_{i\in I} C^\infty(V_i, \mathfrak{k})$ and $\prod_{i\in I} C^\infty(\overline{V}_i, \mathfrak{k})$. Since the isomorphisms

$$C^\infty(P, \mathfrak{k})^K \cong S(\mathrm{Ad}(\mathcal{P})) \cong \mathfrak{g}_{\mathcal{V}}(\mathcal{P}) \cong \mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P}).$$

are all isomorphisms of abstract Lie algebras an isomorphisms of locally convex vector spaces, it follows that they are isomorphisms of topological Lie algebras. ■

As indicated in the introduction to this section, we would like to use smooth sections to pull back elements in $C^{\infty}(P,K)^{K}$ to mappings in $C^{\infty}(M,K)$. Since global sections do not exist in the non-trivial case (by definition), we have to use local sections. This will lead to an isomorphic picture of the gauge group in terms of K-valued mappings, defined on (subsets of) the base M, and transition functions. The following definition and remark will make this precise.

Definition 3.1.5. If $\mathcal{P}$ is a smooth K-principal bundle with compact base M and $\overline{\mathcal{V}}=(\overline{V}_i,\sigma_i)_{i=1,\ldots,n}$ is a smooth closed trivialising system with corresponding transition functions $k_{ij}:\overline{V}_i\cap\overline{V}_j\to K$, then we denote

$$G_{\overline{\mathcal{V}}}(\mathcal{P}):=\left\{(\gamma_i)_{i=1,\ldots,n}\in\prod_{i=1}^{n}C^{\infty}(\overline{V_i},K):\gamma_i(m)=k_{ij}(m)\gamma_j k_{ji}(m)\ \forall m\in\overline{V}_i\cap\overline{V}_j\right\}$$

and turn it into a group with respect to pointwise group operations. ■

Remark 3.1.6. In the situation of Definition 3.1.5, the map

$$\psi:G_{\overline{\mathcal{V}}}(\mathcal{P})\to C^{\infty}(P,K)^{K},\ \psi((\gamma_i)_{i=1,\ldots,n})(p)=k_{\sigma_i}^{-1}(p)\cdot\gamma_i(\pi(p))\cdot k_{\sigma_i}(p)\ \text{ if }\ \pi(p)\in\overline{V}_i \tag{3.1}$$

is an isomorphism of abstract groups, where the map on the right hand side is well-defined because $k_{\sigma_i}(p)=k_{ij}(\pi(p))\cdot k_{\sigma_j}(p)$ and thus

$$\begin{aligned}k_{\sigma_i}^{-1}(p)\cdot\gamma_i(\pi(p))\cdot k_{\sigma_i}(p)&=k_{\sigma_j}(p)^{-1}\cdot\underbrace{k_{ji}(\pi(p))\cdot\gamma_i(\pi(p))\cdot k_{ij}(\pi(p))}_{\gamma_j(\pi(p))}\cdot k_{\sigma_j(p)}\\&=k_{\sigma_j}(p)^{-1}\cdot\gamma_j(\pi(p))\cdot k_{\sigma_j}(p).\end{aligned}$$

In particular, this implies that $\psi((\gamma_i)_{i=1,\ldots,n})$ is smooth. Since for $m\in\overline{V}_i$ the evaluation map $\mathrm{ev}_m:C^{\infty}(\overline{V}_i,K)\to K$ is continuous, $G_{\overline{\mathcal{V}}}(\mathcal{P})$ is a closed subgroup of the Lie group $\prod_{i=1}^{n}C^{\infty}(\overline{V}_i,K)$. ■

Since an infinite-dimensional Lie group may posses closed subgroups which are no Lie groups (cf. [Bo89b, Exercise III.8.2]), the preceding remark does not automatically yield a Lie group structure on $G_{\overline{\mathcal{V}}}(\mathcal{P})$. However, in many situations, it will turn out that $G_{\overline{\mathcal{V}}}(\mathcal{P})$ has a natural Lie group structure.

The following definition encodes the necessary requirement ensuring a Lie group structure on $G_{\overline{\mathcal{V}}}(\mathcal{P})$ that is induced by the natural Lie group structure on $\prod_{i=1}^{n}C^{\infty}(\overline{V}_i,K)$. Since quite different properties of $\mathcal{P}$ will ensure this requirement it seems to be worth extracting it as a condition on $\mathcal{P}$. The name for this requirement will be justified in Corollary 3.1.9.

Definition 3.1.7. Let $\mathcal{P}$ is a smooth principal K-bundle with compact base M and $\overline{\mathcal{V}}=(\overline{V}_i,\sigma_i)_{i=1,\ldots,n}$ be a smooth closed trivialising system. Then we say that

$\mathcal{P}$ has the *property SUB* with respect to $\overline{\mathcal{V}}$ if there exists a convex centred chart $\varphi : W \to W'$ of K such that

$$\varphi_* : G_{\overline{\mathcal{V}}}(\mathcal{P}) \cap \prod_{i=1}^{n} C^\infty(\overline{V}_i, W) \to \mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P}) \cap \prod_{i=1}^{n} C^\infty(\overline{V}_i, W'),\ (\gamma_i)_{i=1,\ldots,n} \mapsto (\varphi \circ \gamma_i)_{i=1,\ldots,n}$$

is bijective. We say that $\mathcal{P}$ has the property SUB if $\mathcal{P}$ has this property with respect to some trivialising system. ■

It should be emphasised that in all relevant cases, known to the author, the bundles have the property SUB, and it is still unclear, whether there are bundles, which do not have this property (cf. Lemma 3.1.13 and Remark 3.1.14). This property now ensures the existence of a natural Lie group structure on $G_{\overline{\mathcal{V}}}(\mathcal{P})$.

Proposition 3.1.8. a) *Let $\mathcal{P}$ be a smooth principal K-bundle with compact base M, which has the property SUB with respect to the smooth closed trivialising system $\overline{\mathcal{V}}$. Then φ_* induces a smooth manifold structure on $G_{\overline{\mathcal{V}}}(\mathcal{P}) \cap \prod_{i=1}^{n} C(\overline{V}_i, W)$. Furthermore, the conditions $i) - iii)$ of Proposition A.1.6 are satisfied such that $G_{\overline{\mathcal{V}}}(\mathcal{P})$ can be turned into a Lie group modelled on $\mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})$.*

b) *In the setting of* a), *the map $\psi : G_{\overline{\mathcal{V}}}(\mathcal{P}) \to C^\infty(P, K)^K$ is an isomorphism of topological groups if $C^\infty(P, K)^K$ is endowed with the subspace topology from $C^\infty(P, K)$.*

c) *In the setting of* a), *we have* $\mathrm{L}(G_{\overline{\mathcal{V}}}(\mathcal{P})) \cong \mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})$.

Proof. a) Set $U := G_{\overline{\mathcal{V}}}(\mathcal{P}) \cap \prod_{i=1}^{n} C(\overline{V}_i, K)$. Since φ_* is bijective by assumption and $\varphi_*(U)$ is open in $\mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})$, it induces a smooth manifold structure on U.

Let $W_0 \subseteq W$ be an open unit neighbourhood with $W_0 \cdot W_0 \subseteq W$ and $W_0^{-1} = W_0$. Then $U_0 := G_{\overline{\mathcal{V}}}(\mathcal{P}) \cap \prod_{i=1}^{n} C^\infty(\overline{V_i}, W_0)$ is an open unit neighbourhood in U with $U_0 \cdot U_0 \subseteq U$ and $U_0 = U_0^{-1}$. Since each $C^\infty(\overline{V_i}, K)$ is a topological group, there exist for each $(\gamma_i)_{i=1,\ldots,n}$ open unit neighbourhoods $U_i \subseteq C^\infty(\overline{V_i}, K)$ with $\gamma_i \cdot U_i \cdot \gamma_i^{-1} \subseteq C^\infty(\overline{V_i}, W)$. Since $C^\infty(\overline{V_i}, W_0)$ is open in $C^\infty(\overline{V_i}, K)$, so is $U_i' := U_i \cap C^\infty(\overline{V_i}, W_0)$. Hence

$$(\gamma_i)_{i=1,\ldots,n} \cdot (G_{\overline{\mathcal{V}}}(\mathcal{P}) \cap (U_1' \times \cdots \times U_n')) \cdot (\gamma_i^{-1})_{i=1,\ldots,n} \subseteq U$$

and conditions $i) - iii)$ of Proposition A.1.6 are satisfied, where the required smoothness properties are consequences of Proposition 2.2.16 and Corollary 2.2.17 (cf. [Gl02b, 3.2]).

b) We show that the map $\psi : G_{\overline{\mathcal{V}}}(\mathcal{P}) \to C^\infty(P, K)^K$ from (3.1) is a homeomorphism. Let $\mathcal{P}|_{\overline{V}_i} =: \mathcal{P}_i$ be the restricted bundle. Since $T^n\overline{V}_i$ is closed in T^nM, we have that $C^\infty(P, K)^K$ is homeomorphic to

$$\widetilde{G}_{\overline{\mathcal{V}}}(\mathcal{P}) := \{(\widetilde{\gamma}_i)_{i=1,\ldots,n} \in \prod_{i=1}^{n} C^\infty(P_i, K)^K : \widetilde{\gamma}_i(p) = \widetilde{\gamma}_j(p) \ \text{ for all } \ p \in \pi^{-1}(\overline{V}_i \cap \overline{V}_j)\}$$

as in Proposition 2.2.9. With respect to this identification, ψ is given by

$$(\gamma_i)_{i=1,\ldots,n} \mapsto (k_{\sigma_i}^{-1} \cdot (\gamma_i \circ \pi) \cdot k_{\sigma_i})_{i=1,\ldots,n}$$

and it thus suffices to show the assertion for trivial bundles. So let $\sigma : M \to P$ be a global section. The map $C^\infty(M,K) \ni f \mapsto f \circ \pi \in C^\infty(P,K)$ is continuous since

$$C^\infty(M,K) \ni f \mapsto T^k(f \circ \pi) = T^k f \circ T^k \pi = (T^k\pi)_*(T^k f) \in C(T^k P, T^k K)$$

is continuous as a composition of a pullback an the map $f \mapsto T^k f$, which defines the topology on $C^\infty(M,K)$. Since conjugation in $C^\infty(P,K)$ is continuous, it follows that φ is continuous. Since the map $f \mapsto f \circ \sigma$ is also continuous (with the same argument), the assertion follows.

c) This follows immediately from $\mathrm{L}(C^\infty(\overline{V_i},K)) \cong C^\infty(\overline{V_i},\mathfrak{k})$ (cf. [Gl02b, Section 3.2]). ■

The next corollary is a mere observation. Since it justifies the name "property SUB", it is made explicit here.

Corollary 3.1.9. *If $\mathcal{P}$ is a smooth principal K-bundle with compact base M, having the property SUB with respect to the smooth closed trivialising system $\overline{\mathcal{V}}$, then $G_{\overline{\mathcal{V}}}(\mathcal{P})$ is a closed subgroup of $\prod_{i=1}^n C^\infty(\overline{V}_i,K)$, which is a Lie group modelled on $\mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})$.* ■

We want to use the isomorphism $\mathrm{Gau}(\mathcal{P}) \cong G_{\overline{\mathcal{V}}}(\mathcal{P})$ to introduce a Lie group structure on $\mathrm{Gau}(\mathcal{P})$. Until now, our construction depends on a particular choice of a trivialising system, but this would be inappropriate for a natural Lie group structure on $\mathrm{Gau}(\mathcal{P})$. We show next that in fact, different choices of trivialising systems lead to isomorphic Lie group structures on $\mathrm{Gau}(\mathcal{P})$.

Proposition 3.1.10. *Let $\mathcal{P}$ be a smooth principal K-bundle over the compact base M. If $\overline{\mathcal{V}} = (\overline{V}_i,\sigma_i)_{i=1,\ldots,n}$ and $\overline{\mathcal{U}} = (\overline{U}_j,\tau_j)_{j=1,\ldots,m}$ are two smooth closed trivialising systems and $\mathcal{P}$ has the property SUB with respect to $\overline{\mathcal{V}}$ and $\overline{\mathcal{U}}$, then $G_{\overline{\mathcal{V}}}(\mathcal{P})$ is isomorphic to $G_{\overline{\mathcal{U}}}(\mathcal{P})$ as a Lie group.*

Proof. First, we note that if the covers underlying $\overline{\mathcal{V}}$ and $\overline{\mathcal{U}}$ are the same, but the sections differ by smooth functions $k_i \in C^\infty(\overline{V}_i,K)$, i.e., $\sigma_i = \tau_i \cdot k_i$, then this induces an automorphism of Lie groups

$$G_{\overline{\mathcal{V}}}(\mathcal{P}) \to G_{\overline{\mathcal{V}}}(\mathcal{P}), \quad (\gamma_i)_{i=1,\ldots,n} \mapsto (k_i^{-1} \cdot \gamma_i \cdot k_i)_{i=1,\ldots,n},$$

because conjugation with k_i^{-1} is an automorphism of $C^\infty(\overline{V}_i,K)$.

Since each two open covers have a common refinement it suffices to show the assertion if one cover is a refinement of the other. So let $V_1,\ldots,V_n$ be a refinement of $U_1,\ldots,U_m$ and let $\{1,\ldots,n\} \ni i \mapsto j(i) \in \{1,\ldots,m\}$ be a function with

$V_i \subseteq U_{j(i)}$. Since different choices of sections lead to automorphisms we may assume that $\sigma_i = \sigma_{j(i)}\big|_{\overline{V}_i}$, implying in particular $k_{ii'}(m) = k_{j(i)j(i')}(m)$. Then the restriction map from Lemma 2.2.20 yields a smooth homomorphism

$$\psi : G_{\overline{\mathcal{U}}}(\mathcal{P}) \to G_{\overline{\mathcal{V}}}(\mathcal{P}), \quad (\gamma_j)_{j\in J} \mapsto (\gamma_{j(i)}\big|_{\overline{V}_i})_{i\in I}.$$

For ψ^{-1} we construct each component $\psi_j^{-1} : G_{\overline{\mathcal{V}}}(\mathcal{P}) \to C^\infty(\overline{U}_j, K)$ separately. The condition that $(\psi_j^{-1})_{j\in J}$ is inverse to ψ is then

$$\psi_j^{-1}((\gamma_i)_{i\in I})\big|_{\overline{V}_i} = \gamma_i \text{ for all } i \text{ with } j = j(i). \tag{3.2}$$

Set $I_j := \{i \in I : \overline{V}_i \subseteq \overline{U}_j\}$ and note that $j(i) = j$ implies $i \in I_j$. Since a change of the sections σ_i induces an automorphism on $G_{\overline{\mathcal{V}}}(\mathcal{P})$ we may assume that $\sigma_i = \sigma_{j(i)}\big|_{\overline{V}_i}$ for each $i \in I_j$. Let $x \in \overline{U}_j \setminus \cup_{i\in I_j} V_i$. Then $x \in V_{i_x}$ for some $i_x \in I$ and thus there exists an open neighbourhood U_x of x such that $\overline{U}_x$ is a manifold with corners, contained in $\overline{U}_j \cap \overline{V}_{i_x}$. Now finitely many $U_{x_1}, \ldots, U_{x_l}$ cover $\overline{U}_j \setminus \cup_{i\in I_j} V_i$ and we set

$$\psi_j^{-1}((\gamma_i)_{i\in I}) = \text{glue}\left((\gamma_i)_{i\in I_j}, \left((k_{ji_{x_k}} \cdot \gamma_{i_{x_k}} \cdot k_{i_{x_k}j})\big|_{U_{x_k}}\right)_{k=1,\ldots,l}\right).$$

Then this defines a smooth map by Proposition 2.2.21 and (3.2) is satisfied because $j(i) = i$ implies $i \in I_j$ ■

We now come to the main result of this section.

Theorem 3.1.11 (Lie group structure on Gau($\mathcal{P}$)). *Let $\mathcal{P}$ be a smooth principal K-bundle over the compact manifold M (possibly with corners). If $\mathcal{P}$ has the property SUB, then* $\mathrm{Gau}(\mathcal{P}) \cong C^\infty(P,K)^K$ *carries a Lie group structure, modelled on $C^\infty(P,\mathfrak{k})^K$. If, moreover, K is locally exponential, then* $\mathrm{Gau}(\mathcal{P})$ *is so.*

Proof. We endow $\mathrm{Gau}(\mathcal{P})$ with the Lie group structure induced from the isomorphisms $\mathrm{Gau}(\mathcal{P}) \cong C^\infty(P,K)^K \cong G_{\overline{\mathcal{V}}}(\mathcal{P})$ for some smooth closed trivialising system $\overline{\mathcal{V}}$. To show that $\mathrm{Gau}(\mathcal{P})$ is locally exponential if K is so we first show that if M is a compact manifold with corners and K has an exponential function, then

$$(\exp_K)_* : C^\infty(M,\mathfrak{k}) \to C^\infty(M,K), \quad \eta \mapsto \exp_K \circ \eta$$

is an exponential function for $C^\infty(M,K)$. For $x \in \mathfrak{k}$ let $\gamma_x \in C^\infty(\mathbb{R},K)$ be the solution of the initial value problem $\gamma(0) = e$, $\gamma'(t) = \gamma(t).x$ (cf. Definition A.1.10). Take $\eta \in C^\infty(M,\mathfrak{k})$. Then

$$\Gamma_\eta : \mathbb{R} \to C^\infty(M,K), \quad (t,m) \mapsto \gamma_{\eta(m)}(t) = \exp_K(t \cdot \eta(m))$$

is a homomorphism of abstract groups. Furthermore, Γ_η is smooth, because it is smooth on a zero neighbourhood of $\mathbb{R}$, for the push-forward of the local inverse of $\exp_K$ provide charts on a unit neighbourhood in $C^\infty(M,K)$. Then

$$\delta^l(\Gamma_\eta)(t) = \Gamma_\eta(t)^{-1}\cdot\Gamma'(t) = \Gamma_\eta(t)^{-1}\cdot\Gamma_\eta(t)\cdot\eta = \eta,$$

thought of as an equation in the Lie group $T\big(C^\infty(M,K)\big)\cong C^\infty(M,\mathfrak{k})\rtimes C^\infty(M,K)$, shows that $\eta\mapsto\Gamma_\eta(1)=\exp_K\circ\gamma$ is an exponential function for $C^\infty(M,K)$. The proof of the preceding lemma yields immediately that

$$\mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})\cap\prod_{i=1}^n C^\infty(\overline{V_i},W')\to G_{\overline{\mathcal{V}}}(\mathcal{P}),\quad (\eta_i)_{i=1,\ldots,n}\mapsto(\exp_K\circ\eta)_{i=1,\ldots,n}$$

is a diffeomorphism and thus $\mathrm{Gau}(\mathcal{P})$ is locally exponential. ■

It remains to elaborate on the arcane property SUB. First we shall see that this property behaves well with respect to refinements of trivialising systems.

Lemma 3.1.12. *Let $\mathcal{P}$ be a smooth principal K-bundle aver the compact base M, and let $\overline{\mathcal{V}}=(\overline{V}_i,\sigma_i)_{i=1,\ldots,n}$ be a smooth closed trivialising system of $\mathcal{P}$. If $\overline{\mathcal{U}}=(\overline{U}_j,\tau_j)_{j=1,\ldots,m}$ is a refinement of $\overline{\mathcal{V}}$, then $\mathcal{P}$ has the property SUB with respect to $\overline{\mathcal{V}}$ if and only if $\mathcal{P}$ has the property SUB with respect to $\overline{\mathcal{U}}$.*

Proof. Let $\{1,\ldots,m\}\ni j\mapsto i(j)\in\{1,\ldots,n\}$ be a map such that $U_j\subseteq V_{i(j)}$ and $\tau_j=\sigma_{i(j)}\big|_{\overline{U}_j}$. Then we have bijective mappings

$$\psi_G: G_{\overline{\mathcal{V}}}(\mathcal{P})\to G_{\overline{\mathcal{U}}}(\mathcal{P}),\quad (\gamma_i)_{i=1,\ldots,n}\mapsto(\gamma_{i(j)}\big|_{\overline{j}})_{j=1,\ldots,m}$$
$$\psi_{\mathfrak{g}}:\mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})\to\mathfrak{g}_{\overline{\mathcal{U}}}(\mathcal{P}),\quad (\eta_i)_{i=1,\ldots,n}\mapsto(\eta_{i(j)}\big|_{\overline{j}})_{j=1,\ldots,m}$$

(cf. Proposition 3.1.10). Now let $\varphi: W\to W'$ be an arbitrary convex centred chart of K and set

$$Q:=G_{\overline{\mathcal{V}}}(\mathcal{P})\cap\prod_{i=1}^n C(\overline{V}_i,W)\qquad \widetilde{Q}:=G_{\overline{\mathcal{U}}}(\mathcal{P})\cap\prod_{i=1}^n C(\overline{U}_i,W)$$
$$Q':=\mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})\cap\prod_{i=1}^n C(\overline{V}_i,W')\qquad \widetilde{Q}':=\mathfrak{g}_{\overline{\mathcal{U}}}(\mathcal{P})\cap\prod_{i=1}^n C(\overline{U}_i,W')$$

Then we have $\psi_G(Q)=\widetilde{Q}$ and $\psi_{\mathfrak{g}}(Q')=\widetilde{Q}'$ and the assertion follows from the commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{\varphi_*} & Q' \\ {\scriptstyle\psi_G}\big\downarrow & & {\scriptstyle\psi_{\mathfrak{g}}}\big\downarrow \\ \widetilde{Q} & \xrightarrow{\varphi_*} & \widetilde{Q}'. \end{array}$$

■

Although it is presently unclear, which bundles have the property SUB and which not, we shall now see that $\mathcal{P}$ has the property SUB in many interesting cases.

Lemma 3.1.13. *Let $\mathcal{P}$ be a smooth principal K-bundle over the compact manifold with corners M. Then $\mathcal{P}$ has the property SUB*

i) with respect to each global smooth trivialising system (M,σ) if $\mathcal{P}$ is trivial,

ii) with respect to each smooth closed trivialising system if K is abelian,

iii) with respect to each smooth closed trivialising system if K is a Banach–Lie group,

iv) with respect to each smooth closed trivialising system if K is locally exponential.

Proof. *i)* If $\mathcal{P}$ is trivial, then there exists a global section $\sigma : M \to P$ and thus $\overline{\mathcal{V}} = (M,\sigma)$ is a trivialising system of $\mathcal{P}$. Then $G_{\overline{\mathcal{V}}}(\mathcal{P}) = C^{\infty}(M,K)$ and φ_* is bijective for any convex centred chart $\varphi : W \to W'$.

ii) If K is abelian, then the conjugation action of K on itself and the adjoint action of K on $\mathfrak{k}$ are trivial. Then a direct verification shows that φ_* is bijective for any trivialising system $\overline{\mathcal{V}}$ and any convex centred chart φ.

iii) If K is a Banach–Lie group, then it is in particular locally exponential (cf. Remark A.1.11) and it thus suffices to show iv).

iv) Let K be locally exponential and $\overline{\mathcal{V}} = (\overline{V}_i,\sigma_i)_{i=1,\ldots,n}$ be a trivialising system. Furthermore, let $W' \subseteq \mathfrak{k}$ be an open zero neighbourhood such that $\exp_K$ restricts to a diffeomorphism on W' and set $W = \exp(W')$ and $\varphi := \exp^{-1} : W \to W'$. Then we have

$$(\gamma_i)_{i=1,\ldots,n} \in G_{\overline{\mathcal{V}}}(\mathcal{P}) \cap \prod_{i=1}^{n} C(\overline{V}_i, W) \Leftrightarrow \varphi_*((\gamma_i)_{i=1,\ldots,n}) \in \mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P}) \cap \prod_{i=1}^{n} C(\overline{V}_i, W'),$$

because $\exp_K(\mathrm{Ad}(k).x) = k \cdot \exp_K(x) \cdot k^{-1}$ holds for all $k \in K$ and $x \in W'$ (cf. Lemma A.1.12). Furthermore, $(\eta_i)_{i=1,\ldots,n} \mapsto (\exp \circ \eta_i)_{i=1,\ldots,n}$ provides an inverse to φ_*.

Since smooth closed trivialising systems always exist by Lemma B.1.13, $\mathcal{P}$ has the property SUB in each of these cases. ■

Remark 3.1.14. The preceding lemma shows that there are different kinds of properties of $\mathcal{P}$ that can ensure the property SUB, i.e., topological in case *i)*, algebraical in case *ii)* and geometrical in case *iv)*. It thus seems to be hard to find a bundle which does *not* have this property. However, a more systematic answer to the question which bundles have this property is not available at the moment. ■

Problem 3.1.15. Is there a smooth principal K-bundle $\mathcal{P}$ over a compact base space M which does not have the property SUB? ■

Lie group structures on the gauge group have already been considered by other authors in similar settings.

Remark 3.1.16. If the structure group K is the group of diffeomorphisms $\mathrm{Diff}(N)$ of some closed compact manifold N, then it does not follow from Lemma 3.1.13 that $\mathcal{P}$ has the property SUB, because $\mathrm{Diff}(N)$ fails to be locally exponential or abelian. However, in this case, $\mathrm{Gau}(\mathcal{P})$ is as a split submanifold of the Lie group $\mathrm{Diff}(P)$, which provides a smooth structure on $\mathrm{Gau}(\mathcal{P})$ [Mi91, Theorem 14.4].

Identifying $\mathrm{Gau}(\mathcal{P})$ with the space of section in the associated bundle $\mathrm{AD}(\mathcal{P})$, where $\mathrm{AD} : K \times K \to K$ is the conjugation action, [OMYK83, Proposition 6.6] also provides a Lie group structure on $\mathrm{Gau}(\mathcal{P})$.

The advantage of Theorem 3.1.11 is, that it provides charts for $\mathrm{Gau}(\mathcal{P})$, which allows us to reduce questions on gauge groups to similar question on mapping groups. This correspondence is crucial for all the following considerations. ■

3.2 Approximation of continuous gauge transformations

As indicated in Appendix A and Section 5.2, obtaining a good knowledge of the (low-dimensional) homotopy groups of an infinite-dimensional Lie group is an important task. The goal of this section is to make the homotopy groups of the gauge group more accessible by reducing their computation to the continuous case, i.e., we shall prove that $\pi_n(\mathrm{Gau}(\mathcal{P}))$ is isomorphic to $\pi_n(\mathrm{Gau}_c(\mathcal{P}))$. Since continuous maps are much more flexible than smooth maps are, this will make the computation of the homotopy groups easier, as explained in Chapter 4.

This chapter was mainly inspired by [Ne02a, Section A.3] and [Hi76, Chapter 2].

We first provide the facts on the group of continuous gauge transformations that we shall need later on.

Remark 3.2.1. Let $\mathcal{P} = (K, \pi : P \to X)$ be a continuous principal K-bundle. Then the same mapping as in the smooth case (cf. Remark 3.1.2) yields an isomorphism

$$\mathrm{Gau}_c(\mathcal{P}) \cong C(P,K)^K := \{\gamma \in C(P,K) : \gamma(p \cdot k) = k^{-1} \cdot \gamma(p) \cdot k \ \forall p \in P, k \in K\},$$

and $C(P,K)^K$ is a topological group as a closed subgroup of $C(P,K)_c$. We equip $\mathrm{Gau}_c(\mathcal{P})$ with the topology defined by this isomorphism.

Let $\overline{\mathcal{V}} = (\overline{V}_i, \sigma_i)_{i\in I}$ be a closed continuous trivialising system of $\mathcal{P}$. Then $\prod_{i\in I} C(\overline{V}_i, K)_c$ is a topological group with

$$G_{c,\overline{\mathcal{V}}}(\mathcal{P}) := \left\{ (\gamma_i)_{i\in I} \in \prod_{i\in I} C(\overline{V_i}, K) : \gamma_i(m) = k_{ij}(m)\cdot\gamma_j(m)\cdot k_{ji}(m) \forall m \in \overline{V}_i \cap \overline{V}_j \right\}$$

as a closed subgroup. Then

$$G_{c,\overline{\mathcal{V}}}(\mathcal{P}) \ni (\gamma_i)_{i\in I} \mapsto \Big(p \mapsto k_{\sigma_i}(p)^{-1}\cdot\gamma_i\left(\pi(p)\right)\cdot k_{\sigma_i}(p) \;\text{ if }\; p\in\pi^{-1}(\overline{V}_i)\Big) \in C(P,K)^K,$$

defines an isomorphism of groups and a straightforward verification shows that this map also defines an isomorphism of topological groups. In exactly the same way one shows that

$$G_{c,\mathcal{V}}(\mathcal{P}) := \left\{ (\gamma_i)_{i\in I} \in \prod_{i\in I} C(V_i, K) : \gamma_i(m) = k_{ij}(m)\cdot\gamma_j(m)\cdot k_{ji}(m) \forall m \in V_i \cap V_j \right\}$$

is also isomorphic to $C(P,K)^K$ as a topological group.

If, in addition, X is compact and $(V_i)_{i\in I}$ also covers X, then there exists a finite subcover $(V_i)_{i=1,\dots,n}$ of X. Since each $C(\overline{V_i}, K)$ is a Lie group [GN07a], the same argumentat as in the proof of Proposition 3.1.8 shows that $C(P,K)^K$, with the subspace-topology from $C(P,K)_c$, can be turned into a Lie group. ■

We collect some concepts and facts from general topology that we shall use throughout this chapter.

Remark 3.2.2. If X is a topological space, then a collection of subsets $(U_i)_{i\in I}$ of X is called *locally finite* if each $x\in X$ has a neighbourhood that has non-empty intersection with only finitely many U_i, and X is called *paracompact* if each open cover has a locally finite refinement. If X is the union of countably many compact subsets, then it is called *σ-compact*, and if each open cover has a countable subcover, it is called *Lindelöf*.

Now let M be a finite-dimensional manifold with corners, which is in particular locally compact and locally connected. For these spaces, [Du66, Theorems XI.7.2+3] imply that M is paracompact if and only if each component is σ-compact or, equivalently, Lindelöf. Furthermore, [Du66, Theorem VIII.2.2] implies that M is normal in each of these cases. ■

Remark 3.2.3. If $\left(\overline{U}_i\right)_{i\in I}$ is a locally finite cover of M by compact sets, then for fixed $i\in I$, the intersection $\overline{U}_i \cap \overline{U}_j$ is non-empty for only finitely many $j\in I$. Indeed, for every $x\in\overline{U}_i$, there is an open neighbourhood U_x of x such that $I_x := \{j\in I : U_x\cap\overline{U}_j \neq\emptyset\}$ is finite. Since $\overline{U}_i$ is compact, it is covered by finitely many of these sets, say by $U_{x_1},\dots,U_{x_n}$. Then $J := I_{x_1}\cup\ldots\cup I_{x_n}$ is the finite set of indices $j\in J$ such that $\overline{U}_i\cap\overline{U}_j$ is non-empty, proving the claim. ■

We now start the business of approximating continuous maps by smooth ones. In the case of functions with values in locally convex spaces, this is quite easy.

Proposition 3.2.4. *If M is a finite-dimensional σ-compact manifold with corners, then for each locally convex space Y the space $C^\infty(M,Y)$ is dense in $C(M,Y)_c$. If $f \in C(M,Y)$ has compact support and U is an open neighbourhood of* $\operatorname{supp}(f)$*, then each neighbourhood of f in $C(M,Y)_c$ contains a smooth function whose support is contained in U.*

Proof. The proof of [Ne02a, Theorem A.3.1] carries over without changes. ∎

Corollary 3.2.5. *If M is a finite-dimensional σ-compact manifold with corners and V is an open subset of the locally convex space Y, then $C^\infty(M,V)$ is dense in $C(M,V)_c$.* ∎

Proof. Since each open subset of $C(M,V)_c$ is also open in $C(M,Y)_c$, this follows immediately from the previous proposition. ∎

We are now aiming for a similar statement for gauge transformations. In order to do so, we need to localise the smoothing process from Proposition 3.2.4. This means to organise an inductive smoothing process in a way that

- at each step, we smooth the function on a region, where it takes values in an open subset of K, which is diffeomorphic to an open convex zero neighbourhood of $\mathfrak{k}$
- when doing so, we should not vary the function in a region, where it is already smooth.

The following lemma provides the tool for this "localised" smoothing process.

Lemma 3.2.6. *Let M be a finite-dimensional σ-compact manifold with corners, Y be a locally convex space, $W \subseteq Y$ be open and convex and let $f : M \to W$ be continuous. If $L \subseteq M$ is closed and $U \subseteq M$ is open such that f is smooth on a neighbourhood of $L \backslash U$, then each neighbourhood of f in $C(M,Y)_c$ contains a continuous map $g : M \to W$, which is smooth on a neighbourhood of L and which equals f on $M \backslash U$.*

Proof. (cf. [Hi76, Theorem 2.5]) Let $A \subseteq M$ be an open set containing $L \backslash U$ such that $f\big|_A$ is smooth. Then $L \backslash A \subseteq U$ is closed in M so that there exists $V \subseteq U$ open with

$$L \backslash A \subseteq V \subseteq \overline{V} \subseteq U$$

Then $\{U, M \backslash \overline{V}\}$ is an open cover of M, and there exists a smooth partition of unity $\{f_1, f_2\}$ subordinated to this cover. Then

$$G_f : C(M,W)_c \to C(M,Y)_c, \;\; G_f(\gamma)(x) = f_1(x)\gamma(x) + f_2(x)f(x)$$

is continuous since $\gamma \mapsto f_1\gamma$ and $f_1\gamma \mapsto f_1\gamma + f_2 f$ are continuous.

If γ is smooth on $A \cup V$ then so is $G_f(\gamma)$, because f_1 and f_2 are smooth, f is smooth on A and $f_2\big|_V \equiv 0$. Note that $L \subseteq A \cup (L \backslash A) \subseteq A \cup V$, so that $A \cup V$ is an open neighbourhood of L. Furthermore we have $G_f(\gamma) = \gamma$ on V and $G_f(\gamma) = f$ on $M \backslash U$. Since $G_f(f) = f$, there is for each open neighbourhood O of f an open neighbourhood O' of f such that $G_f(O') \subseteq O$. By the preceding Corollary there is a smooth function $h \in O'$ such that $g := G_f(h)$ has the desired properties. ∎

We first aim for a generalisation of the previous lemma to functions with values in a locally convex Lie group K. Note that we used a convexity argument in the proof of the previous lemma, showing that the local convexity of K will be crucial for the generalisation to work.

Lemma 3.2.7. *Let M be a finite-dimensional σ-compact manifold with corners, K be a Lie group, $W \subseteq K$ be diffeomorphic to an open convex subset of $\mathfrak{k}$ and $f : M \to W$ be continuous. If $L \subseteq M$ is closed and $U \subseteq M$ is open such that f is smooth on a neighbourhood of $L \backslash U$, then each neighbourhood of f in $C(M, W)_c$ contains a map which is smooth on a neighbourhood of L and which equals f on $M \backslash U$.*

Proof. Let $\varphi : W \to \varphi(W) \subseteq \mathfrak{k}$ be the postulated diffeomorphism. If $\lfloor K_1, V_1 \rfloor \cap \ldots \cap \lfloor K_n, V_n \rfloor$ is an open neighbourhood of $f \in C(M, K)_c$, where we may assume that $V_i \subseteq W$, then $\lfloor K_1, \varphi(V_1) \rfloor \cap \ldots \cap \lfloor K_n, \varphi(V_n) \rfloor$ is an open neighbourhood of $\varphi \circ f$ in $C(M, \varphi(W))_c$. We apply Lemma 3.2.6 to this open neighbourhood to obtain a map h. Then $\varphi^{-1} \circ h$ has the desired properties. ∎

Proposition 3.2.8. *Let M be a connected paracompact finite-dimensional manifold with corners, K be a Lie group and $f \in C(M, K)$. If $L \subseteq M$ is closed and $U \subseteq M$ is open such that f is smooth on a neighbourhood of $L \backslash U$, then each open neighbourhood O of f in $C(M, K)_c$ contains a map g, which is smooth on a neighbourhood of L and equals f on $M \backslash U$.*

Proof. We recall the properties of the topology on M from Remark 3.2.2. If f is smooth on the open neighbourhood A of $L \backslash U$, then there exists an open set $A' \subseteq M$ such that $L \backslash U \subseteq A' \subseteq \overline{A'} \subseteq A$. We choose an open cover $(W_j)_{j \in J}$ of $f(M)$, where each W_j is an open subset of K diffeomorphic to an open zero neighbourhood of $\mathfrak{k}$ and set $V_j := f^{-1}(W_j)$. Since each $x \in M$ has an open neighbourhood $V_{x,j}$ with $\overline{V_{x,j}}$ compact and $\overline{V_{x,j}} \subseteq V_j$ for some $j \in J$, we may redefine the cover $(V_j)_{j \in J}$ such that $\overline{V_j}$ is compact and $f(\overline{V_j}) \subseteq W_j$ for all $j \in J$.

Since M is paracompact, we may assume that the cover $(V_j)_{j \in J}$ is locally finite, and since M is normal, there exists a cover $(V_i')_{i \in I}$ such that for each $i \in I$ there exists a $j \in J$ such that $\overline{V_i'} \subseteq V_j$. Since M is also Lindelöf, we may assume that the latter is countable, i.e., $I = \mathbb{N}^+ := \{1, 2, \ldots\}$. Hence M is also covered by countably many of the V_j and we may thus assume $\overline{V_i'} \subseteq V_i$ and $f(\overline{V_i}) \subseteq W_i$ for each $i \in \mathbb{N}^+$ Furthermore we set $V_0 := \emptyset$ and $V_0' := \emptyset$. Observe that both covers are locally finite by their construction. Define

$$L_i := L \cap \overline{V_i'} \backslash (V_0' \cup \ldots \cup V_{i-1}')$$

which is closed and contained in V_i. Since $L \backslash A' \subseteq U$ we then have $L_i \backslash A' \subseteq V_i \cap U$ and there exist open subsets $U_i \subseteq V_i \cap U$ such that $L_i \backslash A' \subseteq U_i \subseteq \overline{U_i} \subseteq V_i \cap U$. We claim that there exist functions $g_i \in O$, $i \in \mathbb{N}_0$, satisfying

$$\begin{aligned}
&g_i = g_{i-1} \text{ on } M \backslash \overline{U_i} \text{ for } i > 0, \\
&g_i(\overline{V_j}) \subseteq W_j \text{ for all } i, j \in \mathbb{N}_0 \text{ and} \\
&g_i \text{ is smooth on a neighbourhood of } L_0 \cup \ldots \cup L_i \cup \overline{A'}.
\end{aligned}$$

For $i = 0$ we have nothing to show, hence we assume that the g_i are defined for $i < a$. We consider the set

$$Q := \{\gamma \in C(V_a, W_a) : \gamma = g_{a-1} \text{ on } V_a \backslash \overline{U_a}\},$$

which is a closed subspace of $C(V_a, W_a)_c$. Then the map

$$F : Q \to C(M, W_a), \quad F(\gamma)(x) = \begin{cases} \gamma(x) & \text{if } x \in \overline{U_a} \\ g_{a-1}(x) & \text{if } x \in M \backslash \overline{U_a} \end{cases}$$

is continuous since $\overline{U_a}$ is closed. Note that, by induction, $g_{a-1}(V_a) \subseteq W_a$, whence $g_{a-1}|_{V_a} \in Q$. Since F is continuous and $F(g_{a-1}|_{V_a}) = g_{a-1}$, there exists an open set $O' \subseteq C(V_a, W_a)$ containing $g_{a-1}|_{V_a}$ such that $F(O' \cap Q) \subseteq O$.

Since $(V_j)_{j \in \mathbb{N}_0}$ is locally finite and $\overline{V_j}$ is compact, the set $\{j \in \mathbb{N}_0 : \overline{U_a} \cap \overline{V_j} \neq \emptyset\}$ is finite and hence

$$O'' = O' \cap \bigcap_{j \in \mathbb{N}_0} \lfloor \overline{U_a} \cap \overline{V_j}, W_j \rfloor$$

is an open neighbourhood of $g_{a-1}|_{V_a}$ in $C(V_a, W_a)_c$ by induction. We now apply Lemma 3.2.7 with to the manifold with corners V_a, the closed set $L_a' := (L \cap \overline{V_a'}) \cup (\overline{A'} \cap V_a) \subseteq V_a$, the open set $U_a \subseteq V_a$, $g_{a-1}|_{V_a} \in Q \subseteq C(V_a, W_a)$ and the open neighbourhood O'' of $g_{a-1}|_{V_a}$. Due to the construction we have $L_a \backslash U_a \subseteq A' \cap V_a$ and $L \cap \overline{V_a'} \subseteq L_0 \cup \ldots \cup L_a$. Hence we have

$$L_a' \backslash U_a \subseteq (L_0 \cup \ldots \cup L_{a-1} \cup (L_a \backslash U_a)) \cup (\overline{A'} \cap V_a \backslash U_a) \subseteq L_1 \cup \ldots \cup L_{a-1} \cup (\overline{A'} \cap V_a)$$

so that $g_{a-1}|_{V_a}$ is smooth on a neighbourhood of $L_a' \backslash U_a$. We thus obtain a map $h \in O''$ which is smooth on a neighbourhood of L_a' and which coincides with $g_{a-1}|_{V_a}$ on $V_a \backslash U_a \supseteq V_a \backslash \overline{U_a}$, hence is contained in $O'' \cap Q$, and we set $g_a := F(h)$. Since $h(\overline{U_a} \cap \overline{V_j}) \subseteq W_j$ and $g_{a-1}(\overline{V_j}) \subseteq W_j$, we have $F(h)(\overline{V_j}) \subseteq W_j$. Furthermore $F(h)$ inherits the smoothness properties from g_{a-1} on $M \backslash \overline{U_a}$, from h on V_a and since

$L_a \subseteq L \cap \overline{V'_a}$, it has the desired smoothness properties on M. This finishes the construction of the g_i.

We now construct g. First we set $m(x) := \max\{i : x \in \overline{V_i}\}$ and $n(x) := \max\{i : x \in V_i\}$. Then obviously $n(x) \leq m(x)$ and each $x \in M$ has a neighbourhood on which $g_{n(x)}, \ldots, g_{m(x)}$ coincide since $\overline{U_i} \subseteq V_i$ and $g_i = g_{i-1}$ on $M \backslash U_i$. Hence $g(x) := g_{n(x)}(x)$ defines a continuous function on M. If $x \in L$, then $x \in L_0 \cup \ldots \cup L_{n(x)}$ and thus g is smooth on a neighbourhood of x. If $x \in M \backslash U$, then $x \notin U_1 \cup \ldots \cup U_{n(x)}$ and thus $g(x) = f(x)$. ∎

To make the following technical proofs more readable, we first introduce some notation.

Remark 3.2.9. In the remaining section, multiple lower indices on subsets of M always indicate intersections, namely $U_{1\cdots r} := U_1 \cap \ldots \cap U_r$. ∎

The following technical Lemma will make the smoothing process work.

Lemma 3.2.10. *Let M be a manifold with corners that is covered locally finitely by countably many compact sets $\left(\overline{U}_i\right)_{i \in \mathbb{N}}$. Moreover, let $k_{ij} : \overline{U}_{ij} \to K$ be continuous functions into a Lie group K so that $k_{ij} = k_{ji}^{-1}$ holds for all $i, j \in \mathbb{N}$. Then for any convex centred chart $\varphi : W \to \varphi(W)$ of K, each sequence of open unit neighbourhoods $(W'_j)_{j \in N}$ with $W'_j \subseteq W$ and each $\alpha \in \mathbb{N}$, there are φ-convex open unit neighbourhoods $W^{\alpha}_{ij} \subseteq W$ in K for indices $i < j$ and $W^{\alpha}_j \subseteq W'_j$ for $j \in \mathbb{N}$ that satisfy*

$$k_{ji}(x) \cdot (W^{\alpha}_{ij})^{-1} \cdot W^{\alpha}_i \cdot k_{ij}(x) \subseteq W^{\alpha}_j \quad \text{for all} \quad x \in \overline{U}_{ij\alpha} \textit{ and } i < j, \tag{3.3}$$

$$k_{ji}(x) \cdot (W^{\alpha}_{ij})^{-1} \cdot W^{\alpha}_{in} \cdot k_{ij}(x) \subseteq W^{\alpha}_{jn} \quad \text{for all} \quad x \in \overline{U}_{ijn\alpha} \textit{ and } i < j < n \tag{3.4}$$

Proof. Initially, we set $W^{\alpha}_i := W'_i$ for all i, respectively $W^{\alpha}_{ij} := W$ for all $i < j$, disregarding the conditions (3.3) and (3.4). These sets are shrinked later so that they satisfy (3.3) and (3.4).

Our first goal is to satisfy (3.3). We note that the condition in (3.3) becomes trivial if $\overline{U}_{j\alpha}$ is empty, because this implies $\overline{U}_{ij\alpha} = \emptyset$. So we need to consider at most finitely many conditions on (3.3) corresponding to the finitely many $j \in \mathbb{N}$ such that $\overline{U}_{j\alpha} \neq \emptyset$, and we deal with those inductively in decreasing order of j, starting with the maximal such index.

For fixed j and all $i < j$ with $\overline{U}_{ij\alpha} \neq \emptyset$, we describe below how to make the φ-convex unit neighbourhoods W^{α}_{ij} and W^{α}_i on the left hand side smaller so that the corresponding conditions (3.3) are satisfied. Making W^{α}_{ij} and W^{α}_i smaller does not compromise any conditions on $W^{\alpha}_{ij'}$ and $W^{\alpha}_{j'}$ for $j' > j$ that we guaranteed before, because these sets can only appear on the left hand side of such conditions.

To satisfy condition (3.3) for given $i < j$ and W^{α}_j, we note that the function

$$\varphi_{ij} : \overline{U}_{ij\alpha} \times K \times K \to K, \quad (x, k, k') \mapsto k_{ji}(x) \cdot k^{-1} \cdot k' \cdot k_{ij}(x)$$

is continuous and maps all the points (x, e, e) for $x \in \overline{U}_{ij\alpha}$ to e. Hence we may choose open neighbourhoods U_x of x and φ-convex open unit neighbourhoods $W_x \subseteq W_{ij}^{\alpha}$ and $W'_x \subseteq W_i^{\alpha}$ such that $\varphi_{ij}(U_x \times W_x \times W'_x) \subseteq W_j^{\alpha}$. Since $\overline{U}_{ij\alpha}$ is compact, it is covered by finitely many U_x, say by $(U_x)_{x\in F}$ for a finite set $F \subseteq \overline{U}_{ij\alpha}$. Then we replace W_{ij}^{α} and W_i^{α} by their subsets $\bigcap_{x\in F} W_x$ and $\bigcap_{x\in F} W'_x$, respectively, which are φ-convex open unit neighbourhoods in K such that $\varphi_{ij}(\overline{U}_{ij\alpha} \times W_{ij}^{\alpha} \times W_i^{\alpha}) \subseteq W_j^{\alpha}$, in other words, (3.3) is satisfied

Our second goal is to make the sets W_{ij}^{α} also satisfy (3.4), which is non-trivial for the finitely many triples $(i, j, n) \in \mathbb{N}^3$ with $i < j < n$ that satisfy $\overline{U}_{ijn\alpha} \neq \emptyset$. We can argue as above, except for a slightly more complicated order of processing the sets W_{jn}^{α} on the right hand side. Namely, we define the following total order

$$(i, j) < (i', j') \quad :\Leftrightarrow \quad j < j' \text{ or } (j = j' \text{ and } i < i') \tag{3.5}$$

on pairs of real numbers, in particular on pairs of indices (i, j) in $\mathbb{N} \times \mathbb{N}$ with $i < j$. Note that this guarantees $(i, j) < (j, n)$ and $(i, n) < (j, n)$ whenever i, j, n are as in condition (3.4). We process the pairs (j, n) with $\overline{U}_{ijn\alpha} \neq \emptyset$ for some i in descending order, starting with the maximal such pair. At each step, we fix W_{jn}^{α} and make W_{ij}^{α} and W_{in}^{α} smaller for all relevant $i < j$ so that (3.4) is satisfied. This does not violate any conditions (3.3) or (3.4) that we guaranteed earlier in the process, because W_{ij}^{α} and W_{in}^{α} can only appear on the left hand side of such conditions. For the choice of the smaller unit neighbourhoods, we use the continuous function

$$\varphi_{ijn} : \overline{U}_{ijn\alpha} \times K \times K \to K, \quad (x, k, k') \mapsto k_{ji}(x) \cdot k^{-1} \cdot k' \cdot k_{ij}(x)$$

and the compactness of $\overline{U}_{ijn\alpha}$ and argue as before. We thus accomplish our second goal. ■

We are now ready to prove the generalisation of Proposition 3.2.4. This proposition is the first hint that the spaces $C(P, K)^K$ and $C^{\infty}(P, K)^K$ are topologically closely related.

Proposition 3.2.11. *If $\mathcal{P}$ is a smooth principal K-bundle over the connected, paracompact finite-dimensional manifold with corners M, then* $\mathrm{Gau}_c(\mathcal{P})$ *is dense in* $\mathrm{Gau}(\mathcal{P})$.

Proof. Let $(U_j)_{j\in J}$ be a trivialising open cover of M. Proposition 3.3.3 yields locally finite open covers $\big(U_i^{[\lambda]}\big)_{i\in\mathbb{N}}$ of M for every $\lambda \in \{0, \infty\} \cup \big(1 + \frac{1}{3}\mathbb{N}\big)$ such that the closures $\overline{U}_i^{[\lambda]}$ are compact manifolds with corners and

$$\begin{aligned}\overline{U}_i^{[\infty]} \subseteq U_i^{[j+1]} \subseteq \overline{U}_i^{[j+1]} \subseteq U_i^{[j+2/3]} \subseteq \overline{U}_i^{[j+2/3]} \\ \subseteq U_i^{[j+1/3]} \subseteq \overline{U}_i^{[j+1/3]} \subseteq U_i^{[j]} \subseteq U_i^{[0]} \subseteq \overline{U}_i^{[0]} \subseteq U_i\end{aligned}$$

holds for all $i,j\in\mathbb{N}$, where U_i denotes a suitable set of the cover $(U_j)_{j\in J}$ for every $i\in\mathbb{N}$. Furthermore , let

$$k_{ij}:\overline{U}_{ij}^{[0]}\to K$$

be a the transition functions of a fixed cocycle arising from the trivialising cover. By Remark 3.2.1, we may identify $\mathrm{Gau}_c(\mathcal{P})$ with

$$G^{[\infty]}(\mathcal{P}):=\{(\gamma_i)_{i\in\mathbb{N}}\in\prod_{i\in\mathbb{N}}C(\overline{U}_i^{[\infty]},K):\gamma_i(x)=k_{ij}(x)\cdot\gamma_j(x)\cdot k_{ji}(x)\;\forall x\in\overline{U}_{ij}^{[\infty]}\}$$

or with

$$G^{[0]}(\mathcal{P}):=\{(\gamma_i)_{i\in\mathbb{N}}\in\prod_{i\in\mathbb{N}}C(\overline{U}_i^{[0]},K):\gamma_i(x)=k_{ij}(x)\cdot\gamma_j(x)\cdot k_{ji}(x)\;\forall x\in\overline{U}_{ij}^{[0]}\},$$

and each $\gamma=(\gamma_i)_{i\in\mathbb{N}}\in G^{[\infty]}(\mathcal{P})$ is given by the restriction of some uniquely determined element of $G^{[0]}(\mathcal{P})$.

Let $\varphi:W\to\varphi(W)\subseteq\mathfrak{k}$ be a convex centred chart of K. Then a basic open neighbourhood of $(\gamma_i)_{\in\mathbb{N}}$ in $G^{[\infty]}(\mathcal{P})$ is given by

$$\{(\gamma_i')_{i\in\mathbb{N}}\in G^{[\infty]}(\mathcal{P}):(\gamma_i'\cdot\gamma_i^{-1})(\overline{U}_i^{[\infty]})\subseteq W_i\;\text{ for all }\;i\leq m\}\tag{3.6}$$

for open unit neighbourhoods $W_i\subseteq W$. Then

$$k_{ji}(x)\cdot k_{ij}(x)=e\in W_j\;\text{ for all }\;x\in\overline{U}_{ij}^{[0]}\;\text{ and }\;i<j\leq m$$

and a compactness argument as in Lemma 3.2.10 yields open unit neighbourhoods $W_i'\subseteq K$ with

$$k_{ji}(x)\cdot W_i'\cdot k_{ij}(x)\subseteq W_j\;\text{ for all }\;x\in\overline{U}_{ij}^{[0]}\;\text{ and }\;i<j\leq m\tag{3.7}$$

For $i\geq m$, we set $W_i'=W$. We shall inductively construct smooth maps $\widetilde{\gamma}_i:\overline{U}_i^{[0]}\to K$ such that

(a) $\widetilde{\gamma}_j=k_{ji}\cdot\widetilde{\gamma}_i\cdot k_{ij}$ pointwise on $\overline{U}_{ij}^{[j]}$ for all $i<j\in\mathbb{N}$,

(b) $\big(\widetilde{\gamma}_i\cdot\gamma_i^{-1}\big)\big(\overline{U}_{i\alpha}^{[i]}\big)\subseteq W_i^{\alpha}$ for all $i,\alpha\in\mathbb{N}$ and

(c) $(\widetilde{\gamma}_i\cdot\gamma_i^{-1})(\overline{U}_i^{[\infty]})\subseteq W_i$ for all $i\leq m$

are satisfied at each step, where the W_i^{α} are φ-convex unit neighbourhoods provided by Lemma 3.2.10 that we apply to the countable compact cover $\big(\overline{U}_i^{[0]}\big)_{i\in\mathbb{N}}$, to the transition functions k_{ij}, and to $(W_i')_{i\in\mathbb{N}}$. Then $(\widetilde{\gamma}_i|_{\overline{U}_i^{[\infty]}})_{i\in\mathbb{N}}$ is an element of $G^{[\infty]}(\mathcal{P})$, contained in the basic open neighbourhood (3.6) and thus establishes the assertion.

To construct the smooth function $\widetilde{\gamma}_1 : \overline{U}_1^{[0]} \to K$, we apply Proposition 3.2.8 to the continuous map $f := \gamma_1$ on $M := A := U := \overline{U}_1^{[0]}$ and to the open neighbourhood

$$O_1 := \left(\lfloor \overline{U}_1^{[1]}, W_1 \rfloor \cap \bigcap_{\alpha \in \mathbb{N}} \left\lfloor \overline{U}_{1\alpha}^{[1]}, W_1^\alpha \right\rfloor \right) \cdot \gamma_1$$

of γ_1, which is indeed open, since only finitely many $\overline{U}_{1\alpha}^{[1]}$ are non-empty. By construction, $\widetilde{\gamma}_1$ satisfies (b) and (c). To construct the smooth function $\widetilde{\gamma}_j : \overline{U}_j^{[0]} \to K$ inductively for $j > 1$, we need three steps:

- In order to satisfy (b) in the end, we define a map

  $$\widetilde{\gamma}'_j : \bigcup_{i<j} \overline{U}_{ij}^{[j-1]} \to K, \quad \widetilde{\gamma}'_j(x) := k_{ji}(x) \cdot \widetilde{\gamma}_i(x) \cdot k_{ij}(x) \text{ for } x \in \overline{U}_{ij}^{[j-1]}.$$

  If x is an element of both $\overline{U}_{ij}^{[j-1]}$ and $\overline{U}_{i'j}^{[j-1]}$ for $i' < i < j$, condition (a) for $j-1$ and the cocycle condition assert that the so-defined values for $\widetilde{\gamma}'_j(x)$ agree.

- This definition of $\widetilde{\gamma}'_j$, along with properties (a), (b) and (3.3) assert that

  $$\begin{aligned} \varphi_j(x) := \widetilde{\gamma}'_j(x) \cdot \gamma_j(x)^{-1} &= k_{ji}(x) \cdot \widetilde{\gamma}_i(x) \cdot k_{ij}(x) \cdot \gamma_j(x)^{-1} \\ &= k_{ji}(x) \cdot \underbrace{\widetilde{\gamma}_i(x) \cdot \gamma_i(x)^{-1}}_{\in W_i^\alpha} \cdot k_{ij}(x) \in W_j^\alpha \end{aligned}$$

  holds for all $x \in \overline{U}_{ij\alpha}^{[j-1]}$, $i < j$ and α in $\mathbb{N}$. Furthermore, (3.7) ensures that if $j \leq m$, we have

  $$\varphi_j(x) = k_{ji}(x) \cdot \underbrace{\widetilde{\gamma}_i(x) \cdot \gamma_i(x)^{-1}}_{\in W_i^\alpha \subseteq W'_i} \cdot k_{ij}(x) \in W_j$$

  for $x \in \overline{U}_{ij}^{[j-1]}$ and all $i < j$. So we may apply Lemma 3.3.1 to $A := \bigcup_{i<j} \overline{U}_{ij}^{[j-1]}$ and $B := \bigcup_{i<j} \overline{U}_{ij}^{[j-2/3]}$ to fade out φ_j to a continuous map Φ_j on $M := \overline{U}_j^{[0]}$. Then Φ_j coincides with φ_i on B, maps $\overline{U}_{j\alpha}^{[j]}$ into W_j^α and if $j \leq m$ also $\overline{U}_j^{[j-1]}$ into W_j.

- Accordingly, $\Phi_j \cdot \gamma_j$ is an element of the open neighbourhood

  $$O_j := \left(\left\lfloor \overline{U}_j^{[j-1]}, W_j \right\rfloor \cap \bigcap_{\alpha \in \mathbb{N}} \left\lfloor \overline{U}_{j\alpha}^{[j]}, W_j^\alpha \right\rfloor \right) \cdot \gamma_j$$

  of γ_j and is smooth on $\bigcup_{i<j} U_{ij}^{[j-2/3]}$. If we apply Proposition 3.2.8 to $M := A := \overline{U}_j^{[0]}$, $U := M \setminus \bigcup_{i<j} \overline{U}_{ij}^{[j-1/3]}$, O_j, and to $f := \Phi_j \cdot \gamma_j$, then we obtain a smooth map $\widetilde{\gamma}_j : \overline{U}_j^{[0]} \to K$.

The map $\widetilde{\gamma}_j$ satisfies (a), because so does $\widetilde{\gamma}'_j$, with which it coincides on $\bigcup_{i<j} \overline{U}_{ij}^{[j]}$. Moreover, (b) and (c) are satisfied due to the choice of O_j. This concludes the construction. ∎

In combination with the fact that $C^\infty(P,K)^K$ is dense in $C(P,K)^K$, the following fact will provide the isomorphism $\pi_n(C^\infty(P,K)^K) \cong \pi_n(C(P,K)^K)$, which we are aiming for.

Lemma 3.2.12. *Let $\mathcal{P}$ be a smooth principal K-bundle over the compact base M, having the property SUB with respect to the smooth closed trivialising system $\overline{\mathcal{V}} = (\overline{V}_i, \sigma_i)_{i=1,\ldots,n}$ and let $\varphi : W \to W'$ be the corresponding convex centred chart of K (cf. Definition 3.1.7). If $(\gamma_i)_{i=1,\ldots,n} \in G_{\overline{\mathcal{V}}}(\mathcal{P})$ represents an element of $C^\infty(P,K)^K$ (cf. Remark 3.1.6), which is close to identity, in the sense that $\gamma_i(\overline{V_i}) \subseteq W$, then $(\gamma_i)_{i=1,\ldots,n}$ is homotopic to the constant map $(x \mapsto e)_{i=1,\ldots,n}$.*

Proof. Since the map

$$\varphi_* : U := G_{\overline{\mathcal{V}}}(\mathcal{P}) \cap \prod_{i=1}^{n} C^\infty(\overline{V_i}, W) \to \mathfrak{g}(\mathcal{P}), \quad (\gamma'_i)_{i=1,\ldots,n} \mapsto (\varphi \circ \gamma'_i)_{i=1,\ldots,n},$$

is a chart of $G_{\overline{\mathcal{V}}}(\mathcal{P})$ (cf. Proposition 3.1.8) and $\varphi_*(U) \subseteq \mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})$ is convex, the map

$$[0,1] \ni t \mapsto \varphi_*^{-1}\big(t \cdot \varphi_*((\gamma_i)_{i=1,\ldots,n})\big) \in G_{\overline{\mathcal{V}}}(\mathcal{P})$$

defines the desired homotopy. ∎

We finally obtain the main theorem of this section.

Theorem 3.2.13 (Weak homotopy equivalence for Gau($\mathcal{P}$)). *Let $\mathcal{P}$ be a smooth principal K-bundle over the compact manifold M (possibly with corners). If $\mathcal{P}$ has the property SUB, then the natural inclusion $\iota : \mathrm{Gau}(\mathcal{P}) \hookrightarrow \mathrm{Gau}_c(\mathcal{P})$ of smooth into continuous gauge transformations is a weak homotopy equivalence, i.e., the induced mappings $\pi_n(\mathrm{Gau}(\mathcal{P})) \to \pi_n\left(\mathrm{Gau}_c(\mathcal{P})\right)$ are isomorphisms of groups for $n \in \mathbb{N}_0$.*

Proof. We identify $\mathrm{Gau}(\mathcal{P})$ with $C^\infty(P,K)^K$ and $\mathrm{Gau}_c(\mathcal{P})$ with $C(P,K)^K$. To see that $\pi_n(\iota)$ is surjective, consider the continuous principal K-bundle $\mathrm{pr}^*(\mathcal{P})$ obtained form $\mathcal{P}$ by pulling it back along the projection $\mathrm{pr} : \mathbb{S}^n \times M \to M$. Then $\mathrm{pr}^*(\mathcal{P})$ is isomorphic to $(K, \mathrm{id} \times \pi, \mathbb{S}^n \times P, \mathbb{S}^n \times M)$, where K acts trivially on the first factor of $\mathbb{S}^n \times P$. We have with respect to this action $C(\mathrm{pr}^*(P), K)^K \cong C(\mathbb{S}^n \times P, K)^K$ and $C^\infty(\mathrm{pr}^*(P))^K \cong C^\infty(\mathbb{S}^n \times P, K)^K$. The isomorphism $C(\mathbb{S}^n, G_0) \cong C_*(\mathbb{S}^n, G_0) \rtimes G_0 = C_*(\mathbb{S}^n, G) \rtimes G_0$, where $C_*(\mathbb{S}^n, G)$ denotes the space of base-point-preserving maps from $\mathbb{S}^n$ to G, yields

$\pi_n(G) = \pi_0(C_*(\mathbb{S}^n, G)) = \pi_0(C(\mathbb{S}^n, G_0))$ for any topological group G. We thus get a map

$$\pi_n(C^\infty(P,K)^K) = \pi_0(C_*(\mathbb{S}^n, C^\infty(P,K)^K)) = \\ \pi_0(C(\mathbb{S}^n, C^\infty(P,K)^K_0)) \xrightarrow{\eta} \pi_0(C(\mathbb{S}^n, C(P,K)^K_0)),$$

where η is induced by the inclusion $C^\infty(P,K)^K \hookrightarrow C(P,K)^K$.

If $f \in C(\mathbb{S}^n \times P, K)$ represents an element $[F] \in \pi_0(C(\mathbb{S}^n, C(P,K)^K_0))$ (recall $C(P,K)^K \cong G_{c,\mathcal{V}}(\mathcal{P}) \subseteq \prod_{i=1}^n C(V_i,K)$ and $C(\mathbb{S}^n, C(V_i,K)) \cong C(\mathbb{S}^n \times V_i, K)$), then there exists $\widetilde{f} \in C^\infty(\mathbb{S}^n \times P, K)^K$ which is contained in the same connected component of $C(\mathbb{S}^n \times P, K)^K$ as f (cf. Proposition 3.2.11). Since $\widetilde{f}$ is in particular smooth in the second argument, it follows that $\widetilde{f}$ represents an element $\widetilde{F} \in C(\mathbb{S}^n, C^\infty(P,K)^K)$. Since the connected components and the arc components of $C(\mathbb{S}^n \times P, K)^K$ coincide (since it is a Lie group, cf. Remark 3.2.1), there exists a path

$$\tau : [0,1] \to C(\mathbb{S}^n \times P, K)^K_0$$

such that $t \mapsto \tau(t) \cdot f$ is a path connecting f and $\widetilde{f}$. Since $\mathbb{S}^n$ is connected it follows that $C(\mathbb{S}^n \times P, K)^K_0 \cong C(\mathbb{S}^n, C(P,K)^K)_0 \subseteq C(\mathbb{S}^n, C(P,K)^K_0)$. Thus τ represents a path in $C(\mathbb{S}^n, C(P,K)^K_0))$ connecting F and $\widetilde{F}$ whence $[F] = [\widetilde{F}] \in \pi_0(C(\mathbb{S}^n, C(P,K)^K_0))$. That $\pi_n(\iota)$ is injective follows with Lemma 3.2.12 as in [Ne02a, Theorem A.3.7]. ■

This theorem makes the homotopy groups of gauge groups accessible in terms of constructions for continuous mappings. This will be done in Chapter 4.

3.3 Equivalences of principal bundles

This sections presents the results of a joint work with Christoph Müller [MW06]. It develops further the techniques from Section 3.2 and demonstrates the close interplay of bundle theory and topology from a more elementary point of view than homotopy theory, which can be used to obtain the results of this section in the finite-dimensional case (cf. Proposition 3.3.9 for a collection of well-known facts or [Gr58], [To67] and [Gu02] for the case of analytic principal bundles).

The importance of this section is that it shows precisely that there is no difference between continuous and smooth principal bundles, as long as one is only interested in equivalence classes (as one usually is). It thus provides the philosophical background to the interplay between Lie theory and topology encountered in this thesis.

In order to speak of smooth principal bundles one has to consider bundles over manifolds (possibly with corners), whose structure group is a Lie group. The idea of this section is to consider bundles described by transition functions, which are in particular functions with values in Lie groups. Then an appropriate smoothing process, involving the smoothing techniques from Section 3.2, will produce smooth transition functions out of continuous ones and smooth function describing bundle equivalences (or coboundaries) out of continuous ones.

During the mentioned construction process we shall need several technical facts which we provide at first.

Lemma 3.3.1. *Let M be a finite-dimensional paracompact manifold with corners, A and B be closed subsets satisfying $B \subseteq A^0$, $\varphi : W \to \varphi(W)$ be a convex centred chart of a Lie group K modelled on a locally convex space and $f : A \to W$ be a continuous function. Then there is a continuous function $F : M \to W \subseteq K$ with $F|_B = f$ and $F|_{M \setminus A^0} \equiv e$. Moreover, if $W' \subseteq W$ is another φ-convex set containing e, then $f(x) \in W'$ implies $F(x) \in W'$ for each $x \in A$.*

Proof. Since M is paracompact, it is also normal (c.f., Remark 3.2.2). The closed sets $M \setminus A^0$ and B are disjoint by assumption, so the Urysohn Lemma as in [Br93, Theorem I.10.2] yields a continuous function $\lambda : M \to [0,1]$ such that $\lambda|_B \equiv 1$ and $\lambda|_{M \setminus A^0} \equiv 0$. Since $\varphi(W)$ is a convex zero neighbourhood in Y, we have $[0,1] \cdot \varphi(W) \subseteq \varphi(W)$. We use this to define the continuous function

$$f_\lambda : A \to W, \quad x \mapsto \varphi^{-1}\Big(\lambda(x) \cdot \varphi\big(f(x)\big)\Big),$$

that satisfies, by the choice of λ, $f_\lambda|_B = f|_B$ and $f_\lambda|_{\partial A} = e$ because $\partial A \subseteq M \setminus A^0$. So we may extend f_λ to the continuous function

$$F : M \to W, \quad x \mapsto \begin{cases} f_\lambda(x), & \text{if } x \in A \\ e, & \text{if } x \in M \setminus A^0 \end{cases}$$

that satisfies all requirements. ∎

Lemma 3.3.2. *Let W be an open neighbourhood of a point x in $\mathbb{R}^d_+$ (cf. Definition 2.1.7) and $C \subseteq W$ be a compact set containing x. Then there exists an open set V satisfying $x \in C \subseteq V \subseteq \overline{V} \subseteq W$ whose closure $\overline{V}$ is a compact manifold with corners.*

Proof. For every $x = (x_1, \ldots, x_d) \in C$, there is an $\varepsilon_x > 0$ such that

$$B(x, \varepsilon) := [x_1 - \varepsilon_x, x_1 + \varepsilon_x] \times \cdots \times [x_d - \varepsilon_x, x_d + \varepsilon_x] \cap \mathbb{R}^d_+ \tag{3.8}$$

is contained in W. The interiors $V_x := B(x, \varepsilon_x)^0$ in $\mathbb{R}^d_+$ form an open cover of the compact set C, of which we may choose a finite sub collection $(V_{x_i})_{i=1,\ldots,m}$ covering C. The union $V := \bigcup_{i=1}^m V_{x_i}$ satisfies all requirements. In particular, $\overline{V}$ is a compact manifold with corners, because it is a finite union of cubes whose sides are orthogonal to the coordinate axes. ∎

Proposition 3.3.3. *Let M be a finite-dimensional paracompact manifold with corners and $(U_j)_{j\in J}$ be an open cover of M. Then there exist countable open covers $\left(U_i^{[\infty]}\right)_{i\in\mathbb{N}}$ and $\left(U_i^{[0]}\right)_{i\in\mathbb{N}}$ of M such that $\overline{U}_i^{[\infty]} := \overline{U_i^{[\infty]}}$ and $\overline{U}_i^{[0]} := \overline{U_i^{[0]}}$ are compact manifolds with corners, $\overline{U}_i^{[\infty]} \subseteq U_i^{[0]}$ for all $i \in \mathbb{N}$, and such that even the cover $\left(\overline{U}_i^{[0]}\right)_{i\in\mathbb{N}}$ of M by compact sets is locally finite and subordinate to $(U_j)_{j\in J}$.*

In this situation, let L be any countable subset of the open interval $(0,\infty)$. Then for every $\lambda \in L$, there exists a countable, locally finite cover $\left(U_i^{[\lambda]}\right)_{i\in\mathbb{N}}$ of M by open sets whose closures are compact manifolds with corners such that $\overline{U}_i^{[\lambda]} \subseteq U_i^{[\mu]}$ holds whenever $0 \leq \mu < \lambda \leq \infty$.

Proof. For every $x \in M$, we have $x \in U_{j(x)}$ for some $j(x) \in J$. Let (U_x, φ_x) be a chart of M around x such that $\overline{U}_x \subseteq U_{j(x)}$. We can even find an open neighbourhood V_x of x whose closure $\overline{V}_x$ is compact and contained in U_x. Since M is paracompact, the open cover $\left(V_x\right)_{x\in M}$ has a locally finite subordinated cover $(V_i)_{i\in I}$, where $V_i \subseteq V_x$ and $\overline{V_i} \subseteq \overline{V}_x \subseteq U_x$ for suitable $x = x(i)$. Since M is also Lindelöf, we may assume that $I = \mathbb{N}$.

To find suitable covers $U_i^{[\infty]}$ and $U_i^{[0]}$, we are going to enlarge the sets V_i so carefully in two steps that the resulting covers remain locally finite. More precisely, $U_i^{[\infty]}$ and $U_i^{[0]}$ will be defined inductively so that even the family $(V_k^i)_{k\in\mathbb{N}}$ with

$$V_k^i := \begin{cases} \overline{U}_k^{[0]} & \text{for } k \leq i \\ V_k & \text{for } k > i \end{cases}$$

is still a locally finite cover of M for every $i \in \mathbb{N}_0$. We already know this for $i = 0$, because $V_k^0 = V_k$ for all $k \in \mathbb{N}$. For $i > 0$, we proceed by induction.

For every point $y \in \overline{V_i}$, there is an open neighbourhood $V_{i,y}$ of y inside $U_{x(i)}$ whose intersection with just finitely many V_j^{i-1} is non-empty. Under the chart $\varphi_{x(i)}$, this neighbourhood $V_{i,y}$ is mapped to an open neighbourhood of $\varphi_{x(i)}(y)$ in the modelling space $\mathbb{R}_+^d$ of M. There exist real numbers $\varepsilon_0(y) > \varepsilon_\infty(y) > 0$ such that the cubes $B(y, \varepsilon_\infty(y))$ and $B(y, \varepsilon_0(y))$ introduced in (3.8) are compact neighbourhoods of $\varphi_{x(i)}(y)$ contained in $\varphi_{x(i)}(V_{i,y})$. Since $\overline{V_i}$ is compact, it is covered by finitely many sets $V_{i,y}$, say by $\left(V_{i,y}\right)_{y\in Y}$ for a finite subset Y of $\overline{V_i}$. We define the open sets

$$U_i^{[\infty]} := \bigcup_{y\in Y} \varphi_{x(i)}^{-1}\left(B(y, \varepsilon_\infty(y))^0\right) \text{ and } U_i^{[0]} := \bigcup_{y\in Y} \varphi_{x(i)}^{-1}\left(B(y, \varepsilon_0(y))^0\right),$$

whose closures are compact manifolds with corners, because they are a finite union of cubes under the chart $\varphi_{x(i)}$. On the one hand, the construction guarantees

$$V_i \subseteq U_i^{[\infty]} \subseteq \overline{U}_i^{[\infty]} \subseteq U_i^{[0]} \subseteq \overline{U}_i^{[0]} \subseteq \bigcup_{y\in Y} V_{i,y} \subseteq U_{x(i)}.$$

On the other hand, the cover $\left(V_k^i\right)_{k\in\mathbb{N}}$ is locally finite, because it differs from the locally finite cover $\left(V_k^{i-1}\right)_{k\in\mathbb{N}}$ in the single set $V_i^i=\overline{U}_i^{[0]}$.

For a proof of the second claim, we fix an enumeration $\lambda_1,\lambda_2,\ldots$ of L for an inductive construction of the covers. Then for any $n\geq 1$ and $i\in\mathbb{N}$, we apply Lemma 3.3.2 to $C:=\varphi_i\big(\overline{U}_i^{[\overline{\lambda}]}\big)$ and $W:=\varphi_i\big(U_i^{[\underline{\lambda}]}\big)$, where $\overline{\lambda}$ (respectively $\underline{\lambda}$) is the smallest (respectively, largest) element of $\lambda_1,\ldots,\lambda_{n-1}$ larger than (respectively, smaller than) λ_n for $n>1$ and ∞ (respectively, 0) for $n=1$. We get open sets $U_i^{[\lambda_n]}$ such that the condition $\overline{U}_i^{[\lambda]}\subseteq U_i^{[\mu]}$ holds whenever $0\leq\mu<\lambda\leq\infty$ are elements in $\{\lambda_1,\ldots,\lambda_n\}$, and eventually in L. This completes the proof. ■

In order to make the technical constructions more readable we introduce the following abbreviation.

Remark 3.3.4. In the remaining section, multiple lower indices on subsets of M always indicate intersections, namely $U_{1\ldots r}:=U_1\cap\ldots\cap U_r$. ■

The following two theorems require to construct principal bundles and/or equivalences between them, by constructing inductively cocycles and representatives of equivalences. In these constructions, every new transition function (respectively, every new local representative of an equivalence)

- is already determined by cocycle conditions (respectively, by compatibility conditions) on a subset of its domain,
- from which it will be "faded out" to a continuous function on the whole domain
- and smoothed, if necessary.

In each such step, we need a safety margin to modify the functions without compromising previous achievements too much, and these safety margins are the nested open covers provided by Proposition 3.3.3. In order to "fade out" appropriately, we need to make sure that the values of the corresponding functions stay in certain unit neighbourhoods of the structure group. This is achieved with the data from Lemma 3.2.10.

During the construction we will violate the cocycle and compatibility condition $k_{ij}=k_{in}\cdot k_{nj}$ and $f_i=k_{ij}\cdot f_j\cdot k_{ji}$. But we will alway assure that these conditions are still satisfied on the open cover $(U_i^{[\infty]})_{i\in\mathbb{N}}$. This suffices to determine smooth cocycles and smooth bundle equivalences completely.

Theorem 3.3.5 (Smoothing continuous principal bundles). *Let K be a Lie group modelled on a locally convex space, M be a finite-dimensional paracompact manifold (possibly with corners) and $\mathcal{P}$ be a continuous principal K-bundle over M. Then there exists a smooth principal K-bundle $\widetilde{\mathcal{P}}$ over M and a continuous bundle equivalence $\Omega:\mathcal{P}\to\widetilde{\mathcal{P}}$.*

Proof. We assume that the continuous bundle $\mathcal{P}$ is given by $\mathcal{P}_k$ as in Remark B.1.7, where $(U_j)_{j\in J}$ is a locally trivial cover of M and $k_{ij} : U_{ij} \to K$ are continuous transition functions that satisfy the cocycle condition $k_{ij} \cdot k_{jn} = k_{in}$ pointwise on U_{ijn}.

Proposition 3.3.3 yields open covers $\big(U_i^{[\infty]}\big)_{i\in\mathbb{N}}$ and $\big(U_i^{[0]}\big)_{i\in\mathbb{N}}$ of M subordinate to $(U_j)_{j\in J}$ with $\overline{U}_i^{[\infty]} \subseteq U_i^{[0]}$ for all $i \in \mathbb{N}$. For every $i \in \mathbb{N}$, we denote by U_i an open set of the cover $(U_j)_{j\in J}$ that contains $U_i^{[0]}$ and observe that $(U_i)_{i\in\mathbb{N}}$ is still a locally trivial open cover of M. In our construction, we need open covers not only for pairs $(j,n) \in \mathbb{N}\times\mathbb{N}$ with $j < n$, but also for pairs $(j-1/3, n)$, $(j-2/3, n)$ in-between and (n,n) to enable continuous extensions and smoothing. The function

$$\lambda : \left\{(j,n) \in \frac{1}{3}\mathbb{N}_0 \times \mathbb{N} : j \le n\right\} \to [0,\infty), \quad \lambda(j,n) = \frac{n(n-1)}{2} + j,$$

is tailored to map the pairs $(0,1),(1,1),(1,2),(2,2),(1,3),(2,3),(3,3),(1,4),\ldots$ to the integers $0,1,2,\ldots$, respectively, and the other pairs in-between. If we apply the second part of Proposition 3.3.3 to the countable subset $L := (\operatorname{im}\lambda)\setminus\{0\}$ of $(0,\infty)$, we get open sets $U_i^{[jn]} := U_i^{[\lambda(j,n)]}$ for all pairs (j,n) in the domain of λ such that $\big(\overline{U}_i^{[jn]}\big)_{i\in\mathbb{N}}$ are again locally finite covers. We note that $(j,n) < (j',n')$ in the sense of (3.5) implies $\overline{U}_i^{[j'n']} \subseteq U_i^{[jn]}$.

Let $\varphi : W \to \varphi(W)$ be an arbitrary convex centred chart of K and consider the countable compact cover $\big(\overline{U}_i^{[0]}\big)_{i\in\mathbb{N}}$ of M and the restrictions $k_{ij}|_{\overline{U}_{ij}^{[0]}}$ of the continuous transition functions to the corresponding intersections. Then Lemma 3.2.10 yields open φ-convex unit neighbourhoods W_{ij}^{α} and W_i^{α} with the corresponding properties.

Our first goal is the construction of smooth maps $\widetilde{k}_{ij} : \overline{U}_{ij}^{[0]} \to K$ that satisfy the cocycle condition on the open cover $\big(U_i^{[\infty]}\big)_{i\in\mathbb{N}}$ of M, which uniquely determines a smooth principal K-bundle $\widetilde{\mathcal{P}} = \mathcal{P}_{\widetilde{k}}$ by Remarks B.1.7 and B.1.12. These maps $\widetilde{k}_{ij}$ will be constructed step-by-step in increasing order with respect to (3.5), starting with the minimal index $(1,2)$. At all times during the construction, the conditions

(a) $\widetilde{k}_{jn} = \widetilde{k}_{ji}\cdot\widetilde{k}_{in}$ pointwise on $\overline{U}_{ijn}^{[jn]}$ for $i < j < n$ in $\mathbb{N}$ and

(b) $\big(\widetilde{k}_{jn}\cdot k_{nj}\big)\big(\overline{U}_{jn\alpha}^{[jn]}\big) \subseteq W_{jn}^{\alpha}$ for all $j < n$ and α in $\mathbb{N}$,

will be satisfied whenever all $\widetilde{k}_{ij}$ involved have already been constructed. We are now going to construct the smooth maps $\widetilde{k}_{jn}$ for indices $j < n$ in $\mathbb{N}$ (and implicitly $\widetilde{k}_{nj}$ as $\widetilde{k}_{nj}(x) := \widetilde{k}_{jn}(x)^{-1}$), assuming that this has already been done for pairs of indices smaller than (j,n).

- To satisfy all relevant cocycle conditions, we start with

$$\widetilde{k}'_{jn} : \bigcup_{i<j} \overline{U}_{ijn}^{[j-1,n]} \to K, \quad \widetilde{k}'_{jn}(x) := \widetilde{k}_{ji}(x)\cdot\widetilde{k}_{in}(x) \text{ for } x \in \overline{U}_{ijn}^{[j-1,n]}.$$

This smooth function is well-defined, because the cocycle conditions (a) for lower indices assert that for any indices $i' < i < j$ and any point $x \in \overline{U}_{i'jn}^{[j-1,n]} \cap \overline{U}_{ijn}^{[j-1,n]}$, we have

$$\widetilde{k}_{ji'}(x) \cdot \widetilde{k}_{i'n}(x) = \widetilde{k}_{ji'}(x) \cdot \widetilde{k}_{i'i}(x) \cdot \widetilde{k}_{ii'}(x) \cdot \widetilde{k}_{in}(x) = \widetilde{k}_{ji}(x) \cdot \widetilde{k}_{in}(x),$$

because $\overline{U}_{i'ijn}^{[j-1,n]}$ is contained in both $\overline{U}_{i'ij}^{[ij]}$ and $\overline{U}_{i'in}^{[in]}$.

- Next, we want to extend the smooth map $\widetilde{k}'_{jn}$ on $\bigcup_{i<j} \overline{U}_{ijn}^{[j-1,n]}$ to a continuous map k'_{jn} on $\overline{U}_{jn}^{[0]}$ without compromising the cocycle conditions too much. To do this, we consider the function $\varphi_{jn} := \widetilde{k}'_{jn} k_{nj} : \bigcup_{i<j} \overline{U}_{ijn}^{[j-1,n]} \to K$. For all $i < j$, $\alpha \in \mathbb{N}$ and $x \in \overline{U}_{ijn\alpha}^{[j-1,n]}$, conditions (b) above and (3.4) of Lemma 3.2.10 imply

$$\begin{aligned} \varphi_{jn}(x) = (\widetilde{k}'_{jn} k_{nj})(x) &= k_{ji}(x) \cdot \big(\underbrace{(\widetilde{k}_{ij} \cdot k_{ji})(x)}_{\in W_{ij}^{\alpha}}\big)^{-1} \cdot \underbrace{(\widetilde{k}_{in} \cdot k_{ni})(x)}_{\in W_{in}^{\alpha}} \cdot k_{ij}(x) \\ &\in k_{ji}(x) \cdot (W_{ij}^{\alpha})^{-1} \cdot W_{in}^{\alpha} \cdot k_{ij}(x) \subseteq W_{jn}^{\alpha}, \end{aligned}$$

because $\overline{U}_{ijn\alpha}^{[j-1,n]}$ is contained in both $\overline{U}_{ij\alpha}^{[ij]}$ and $\overline{U}_{in\alpha}^{[in]}$. Since the values of φ_{jn} are contained in particular in the unit neighbourhood W, we may apply Lemma 3.3.1 to $M := \overline{U}_{jn}^{[0]}$ and its subsets $A := \bigcup_{i<j} \overline{U}_{ijn}^{[j-1,n]}$ and $B := \bigcup_{i<j} \overline{U}_{ijn}^{[j-2/3,n]}$. It yields a continuous function $\Phi_{jn} : \overline{U}_{jn}^{[0]} \to W$ that coincides with φ_{jn} on B, is the identity outside A, and satisfies $\Phi_{jn}(x) \in W_{jn}^{\alpha}$ for all $x \in \overline{U}_{jn\alpha}^{[j-1,n]}$. We define $k'_{jn} : \overline{U}_{jn}^{[0]} \to K$ by $k'_{jn} := \Phi_{jn} k_{jn}$ and note that k'_{jn} coincides with the smooth function $\widetilde{k}'_{jn}$ on B and with k_{jn} outside A.

- We finally get the smooth map $\widetilde{k}_{jn} : \overline{U}_{jn}^{[0]} \to K$ that we are looking for if we apply Proposition 3.2.8 to the function k'_{jn} on $M := A := \overline{U}_{jn}^{[0]}$, to the open complement U of $\bigcup_{i<j} \overline{U}_{ijn}^{[j-1/3,n]}$ in M, and to the neighbourhood

$$O_{jn} := \left(\bigcap_{\alpha \in \mathbb{N}} \left\lfloor \overline{U}_{jn\alpha}^{[jn]}, W_{jn}^{\alpha} \right\rfloor \right) \cdot k_{jn}$$

of both k_{jn} and k'_{jn}, where $k'_{jn} \in O_{jn}$ follows from firstly $\Phi_{jn}(x) \in W_{jn}^{\alpha}$ and secondly $k'_{jn}(x) = \Phi_{jn}(x) \cdot k_{jn}(x) \in W_{jn}^{\alpha} \cdot k_{jn}(x)$ for all $x \in \overline{U}_{jn\alpha}^{[jn]}$. Note that O_{jn} is really open, because Remark 3.2.3 asserts that just finitely many of the sets $\overline{U}_{jn\alpha}^{[jn]}$ for fixed $\alpha \in \mathbb{N}$ are non-empty and may influence the intersection.

  By the choice of U, the result $\widetilde{k}_{jn}$ coincides with both k'_{jn} and $\widetilde{k}'_{jn}$ on $\bigcup_{i<j} \overline{U}_{ijn}^{[jn]}$, so it satisfies the cocycle conditions (a). It also satisfies (b) by the choice of O_{jn}.

This concludes the construction of the smooth principal K-bundle $\widetilde{\mathcal{P}}$. We use the same covers of M and unit neighbourhoods in K for the construction of continuous functions $f_i : \overline{U}_i^{[0]} \to K$ such that

(c) $f_n = \widetilde{k}_{nj} \cdot f_j \cdot k_{jn}$ pointwise on $\overline{U}_{jn}^{[nn]}$ for $j < n$ in $\mathbb{N}$ and

(d) $f_n\big(\overline{U}_{n\alpha}^{[0]}\big) \subseteq W_n^\alpha$ for $\alpha, n \in \mathbb{N}$.

Then Remark B.1.9 tells us that the restriction of the maps f_i to the sets $U_i^{[\infty]}$ of the open cover is the local description of a bundle equivalence $\Omega : \mathcal{P} \to \widetilde{\mathcal{P}}$ that we are looking for. Indeed, all the sets $\overline{U}_{jn}^{[nn]}$ of condition (c) contain the corresponding sets $U_{jn}^{[\infty]}$ of the open cover.

We start with the constant function $f_1 \equiv e$, which clearly satisfies condition (d). Then we construct f_n for $n > 1$ inductively as follows:

- To satisfy condition (c), we start with
$$f_n' : \bigcup_{j<n} \overline{U}_{jn}^{[jn]} \to K, \quad f_n'(x) = \widetilde{k}_{nj}(x) \cdot f_j(x) \cdot k_{jn}(x) \text{ for } x \in \overline{U}_{jn}^{[jn]}.$$
This continuous function is well-defined, because the conditions (c) for f_j on $\overline{U}_{j'jn}^{[jn]} \subseteq \overline{U}_{j'j}^{[jj]}$ and (a) for $j' < j < n$ on $\overline{U}_{j'jn}^{[jn]}$ guarantee that
$$\begin{aligned}\widetilde{k}_{nj}(x) \cdot f_j(x) \cdot k_{jn}(x) &= \widetilde{k}_{nj}(x) \cdot \widetilde{k}_{jj'}(x) \cdot f_{j'}(x) \cdot k_{j'j}(x) \cdot k_{jn}(x) \\ &= \widetilde{k}_{nj'}(x) \cdot f_{j'}(x) \cdot k_{j'n}(x)\end{aligned}$$
holds for all $x \in \overline{U}_{j'jn}^{[jn]}$.

- To apply Lemma 3.3.1, we need to know something about the values of f_n'. For arbitrary $\alpha \in \mathbb{N}$ and $x \in \overline{U}_{jn\alpha}^{[jn]}$, conditions (b), (d), and (3.3) of Lemma 3.2.10 imply
$$\begin{aligned}f_n'(x) = \widetilde{k}_{nj}(x) \cdot f_j(x) \cdot k_{jn}(x) &= k_{nj}(x) \cdot \big(\widetilde{k}_{jn}(x) \cdot k_{nj}(x)\big)^{-1} \cdot f_j(x) \cdot k_{jn}(x) \\ &\in k_{nj}(x) \cdot \big(W_{jn}^\alpha\big)^{-1} \cdot W_j^\alpha \cdot k_{jn}(x) \subseteq W_n^\alpha,\end{aligned}$$
so that the values of f_n' are, altogether, contained in the unit neighbourhood W of K. If we apply Lemma 3.3.1 to $M := \overline{U}_n^{[0]}$, to f_n' on $A := \bigcup_{j<n} \overline{U}_{jn}^{[jn]}$ and to the smaller set $B := \bigcup_{j<n} \overline{U}_{jn}^{[nn]}$, then we get a continuous function $f_n : \overline{U}_n^{[0]} \to W$ that satisfies both (c) and (d).

This concludes the construction of the bundle equivalence. ∎

Theorem 3.3.6 (Smoothing continuous bundle equivalences). *Let K be a Lie group modelled on a locally convex space, M be a finite-dimensional paracompact manifold (possibly with corners) and $\mathcal{P}$ and $\mathcal{P}'$ be two smooth principal K-bundles over M. If there exists a continuous bundle equivalence $\Omega : P \to P'$, then there exists a smooth bundle equivalence $\widetilde{\Omega} : P \to P'$.*

Proof. Let $(U_j)_{j\in J}$ be an open cover of M that is locally trivial for both bundles $\mathcal{P}$ and $\mathcal{P}'$. Proposition 3.3.3 yields locally finite open covers $\big(U_i^{[\lambda]}\big)_{i\in\mathbb{N}}$ of M for every $\lambda \in \{0,\infty\} \cup \big(1+\frac{1}{3}\mathbb{N}\big)$ such that the closures $\overline{U}_i^{[\lambda]}$ are compact manifold with corners and

$$\overline{U}_i^{[\infty]} \subseteq U_i^{[j+1]} \subseteq \overline{U}_i^{[j+1]} \subseteq U_i^{[j+2/3]} \subseteq \overline{U}_i^{[j+2/3]} \\ \subseteq U_i^{[j+1/3]} \subseteq \overline{U}_i^{[j+1/3]} \subseteq U_i^{[j]} \subseteq U_i^{[0]} \subseteq \overline{U}_i^{[0]} \subseteq U_i$$

holds for all $i,j \in \mathbb{N}$, where U_i denotes a suitable set of the cover $(U_j)_{j\in J}$ for every $i \in \mathbb{N}$. According to Remarks B.1.7 and B.1.12, we may then describe the smooth bundles $\mathcal{P}$ and $\mathcal{P}'$ by smooth transition functions $k = (k_{ij})_{i,j\in\mathbb{N}}$ and $k' = (k'_{ij})_{i,j\in\mathbb{N}}$ on the open cover $(U_i)_{i\in\mathbb{N}}$, equivalently, by their restrictions to any open cover $\big(U_i^{[\lambda]}\big)_{i\in\mathbb{N}}$ from above. In these local descriptions of the bundles, the bundle equivalence Ω can, as in Remark B.1.9, be seen as a family $f_i : U_i \to K$ of continuous maps for $i \in \mathbb{N}$ that satisfy

$$f_j(x) = k'_{ji}(x) \cdot f_i(x) \cdot k_{ij}(x) \text{ for all } i,j \in \mathbb{N} \text{ and } x \in U_{ij}. \tag{3.9}$$

We shall inductively construct smooth maps $\widetilde{f}_i : \overline{U}_i^{[0]} \to K$ such that

(a) $\widetilde{f}_j = k'_{ji} \cdot \widetilde{f}_i \cdot k_{ij}$ pointwise on $\overline{U}_{ij}^{[j]}$ for all $i < j$ in $\mathbb{N}$ and

(b) $\big(\widetilde{f}_i \cdot f_i^{-1}\big)\big(\overline{U}_{i\alpha}^{[i]}\big) \subseteq W_i^{\alpha}$ for all $i,\alpha \in \mathbb{N}$

are satisfied at each step, where the W_i^{α} are φ-convex unit neighbourhoods provided by Lemma 3.2.10 that we apply to the countable compact cover $\big(\overline{U}_i^{[0]}\big)_{i\in\mathbb{N}}$, to the transition functions k'_{ij}, and to a convex centred chart $\varphi : W \to \varphi(W)$ of K (we do not need the W_{ij}^{α} in this proof). These maps $\widetilde{f}_i$ describe a smooth bundle equivalence between $\mathcal{P}$ and $\mathcal{P}'$ when restricted to the open cover $\big(U_i^{[\infty]}\big)_{i\in\mathbb{N}}$, because (a) asserts that $\widetilde{f}_j = k'_{ji} \cdot \widetilde{f}'_i \cdot k_{ij}$ is satisfied on $U_{ij}^{[\infty]}$ for all $i < j$, in particular.

To construct the smooth function $\widetilde{f}_1 : \overline{U}_1^{[0]} \to K$, we apply Proposition 3.2.8 to the continuous map $f := f_1$ on $M := A := U := \overline{U}_1^{[0]}$ and to the open neighbourhood

$$O_1 := \bigcap_{\alpha\in\mathbb{N}} \Big\lfloor \overline{U}_{1\alpha}^{[0]}, W_1^{\alpha} \Big\rfloor \cdot f_1$$

of f_1, which is indeed open, since only finitely many $\overline{U}^{[0]}_{1\alpha}$ are non-empty by Remark 3.2.3. By construction, $\widetilde{f}_1$ satisfies (b). To construct the smooth function $\widetilde{f}_j : \overline{U}^{[0]}_j \to K$ inductively for $j > 1$, we need the usual three steps:

- In order to satisfy (b) in the end, we define a continuous map
$$\widetilde{f}'_j : \bigcup_{i<j} \overline{U}^{[j-1]}_{ij} \to K, \quad \widetilde{f}'_j(x) := k'_{ji}(x) \cdot \widetilde{f}_i(x) \cdot k_{ij}(x) \text{ for } x \in \overline{U}^{[j-1]}_{ij}.$$
If x is an element of both $\overline{U}^{[j-1]}_{ij}$ and $\overline{U}^{[j-1]}_{i'j}$ for $i' < i < j$, condition (a) for $j-1$ and the cocycle conditions of both k and k' assert that the so-defined values for $\widetilde{f}'_j(x)$ agree.

- This definition of $\widetilde{f}'_j$, along with (3.9) and property (3.3) in Lemma 3.2.10 assert that
$$\begin{aligned}\varphi_j(x) := \widetilde{f}'_j(x) \cdot f_j(x)^{-1} = k'_{ji}(x) \cdot \widetilde{f}_i(x) \cdot k_{ij}(x) \cdot f_j(x)^{-1} \\ = k'_{ji}(x) \cdot \underbrace{\widetilde{f}_i(x) \cdot f_i(x)^{-1}}_{\in W^\alpha_i} \cdot k'_{ij}(x) \in W^\alpha_j\end{aligned}$$
holds for all $x \in \overline{U}^{[j-1]}_{ij\alpha}$, $i < j$ and α in $\mathbb{N}$. So we may apply Lemma 3.3.1 to $A := \bigcup_{i<j} \overline{U}^{[j-1]}_{ij}$ and $B := \bigcup_{i<j} \overline{U}^{[j-2/3]}_{ij}$ to fade out φ_j to a continuous map Φ_j on $M := \overline{U}^{[0]}_j$. Then Φ_j coincides with φ_i on B and maps $\overline{U}^{[j]}_{j\alpha}$ into W^α_j.

- Accordingly, $\Phi_j \cdot f_j$ is an element of the open (due to Remark 3.2.3) neighbourhood
$$O_j := \bigcap_{\alpha \in \mathbb{N}} \left\lfloor \overline{U}^{[j]}_{j\alpha}, W^\alpha_j \right\rfloor \cdot f_j$$
of f_j and is smooth on $\bigcup_{i<j} U^{[j-2/3]}_{ij}$. If we apply Proposition 3.2.8 to $M := A := \overline{U}^{[0]}_j$, $U := M \setminus \bigcup_{i<j} \overline{U}^{[j-1/3]}_{ij}$, O_j, and to $f := \Phi_j \cdot f_j$, then we obtain a smooth map $\widetilde{f}_j : \overline{U}^{[0]}_j \to K$.

The map $\widetilde{f}_j$ satisfies (a), because so does $\widetilde{f}'_j$, with which it coincides on $\bigcup_{i<j} \overline{U}^{[j]}_{ij}$. Moreover, (b) is satisfied due to the choice of O_j. This concludes the construction.■

In the remaining section, we explain the relations of the preceding theorems to classical bundle theory, non-abelian Čech cohomology and to twisted K-theory. The following lemma on smoothing homotopies will provide the tool we need when smoothing principal bundles, which are given in terms of classifying maps.

Lemma 3.3.7. *([KM02], [Wo06, Corollary 12]) Let M be a manifold with corners and N be a locally convex manifold. If $f : M \to N$ is continuous, then there exists a continuous map $F : [0,1] \times M \to N$ such that $F(0,x) = f(x)$ and*

$F(1,\cdot): M \to N$ is smooth. Furthermore, if $f,g: M \to N$ are smooth and there exists a continuous homotopy between f and g, then there exists a smooth homotopy between f and g.

Lemma 3.3.8. *If K is a compact Lie group, then it has a smooth universal bundle $EK \to BK$ with a smooth classifying space BK, which is in general infinite-dimensional.*

Proof. Let $O_k \subseteq \mathrm{GL}_k(\mathbb{R})$ denote the orthogonal group. If k is sufficiently large, then we may identify K with a subgroup of O_k, and from [St51, Theorem 19.6] we get the following formulae:

$$\begin{aligned} EK &= \lim_{\to} O_n/\big(O_{n-k} \times \mathrm{id}_{\mathbb{R}^k}\big), \\ BK &= \lim_{\to} O_n/\big(O_{n-k} \times K\big). \end{aligned}$$

Thus EK and BK are smooth manifolds by [Gl05, Theorem 3.1] as a direct limit of finite-dimensional manifolds. Since the action of K is smooth, it follows that $EK \to BK$ is a smooth K-principal bundle. ■

Proposition 3.3.9. *If $\mathcal{P}$ is a continuous principal K-bundle over M, K is a finite-dimensional Lie group and M is a finite-dimensional manifold with corners, then there exists a smooth principal K-bundle which is continuously equivalent to $\mathcal{P}$. Moreover, two smooth principal K-bundles over M are smoothly equivalent if and only if they are continuously equivalent.*

Proof. Let C be a maximal compact subgroup of K. Since K/C is contractible, there exists a C-reduction of $\mathcal{P}$, i.e., we may choose a locally trivial open cover $(U_i)_{i\in I}$ with transition functions k_{ij} that take values in C. They define a continuous principal C-bundle which is given by a classifying map $f: M \to BC$.

By Lemma 3.3.7, f is homotopic to some smooth map $\widetilde{f}: M \to BC$ which in turn determines a smooth principal C-bundle $\widetilde{\mathcal{P}}$ over M given by smooth transition functions $\widetilde{k}_{ij}$. Furthermore, since f and $\widetilde{f}$ are homotopic, $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are equivalent, and we thus have a continuous bundle equivalence given by continuous mappings $f_i: U_i \to C$. The claim follows if we regard k_{ij}, $\widetilde{k}_{ij}$ and f_i as mappings into K.

Since smooth bundles yield smooth classifying maps and smooth homotopies of classifying maps yield smooth bundle equivalences (all the constructions in the topological setting depend only on partitions of unity which we can assume to be smooth here), the second claim is also immediate. ■

We now reformulate Theorem 3.3.5 and Theorem 3.3.6 in terms of non-abelian Čech cohomology.

Remark 3.3.10. Let M be a paracompact topological space with an open cover $\mathcal{U} = (U_i)_{i\in I}$ and A be an abelian topological group. Then for $n \geq 0$, an *n-cochain* f is a collection of continuous functions $f_{i_1 \ldots i_{n+1}} : U_{i_1\ldots i_{n+1}} \to A$, and we denote the set of n-cochains by $C^n(\mathcal{U}, A)$ and set it to $\{0\}$ if $n < 0$. We then define the boundary operator

$$\partial_n : C^n(\mathcal{U}, A) \to C^{n+1}(\mathcal{U}, A), \quad \partial(f)_{i_0 i_1\ldots i_{n+1}} = \sum_{k=0}^{n} (-1)^k f_{i_0\ldots \widehat{i_k}\ldots i_{n+1}},$$

where $\widehat{i_k}$ means that we omit the index i_k. Then $\partial_{n+1} \circ \partial_n = 0$, and we define

$$\check{H}^n_c(\mathcal{U}, A) := \ker(\partial_n)/\operatorname{im}(\partial_{n-1}) \quad \text{and} \quad \check{H}^n_c(M, A) := \lim_{\to} \check{H}^n_c(\mathcal{U}, A). \tag{3.10}$$

■

The group $\check{H}^1(M, A)$ is the *n-th continuous Čech cohomology.* If, in addition, M is a smooth manifold with or without corners and A is a Lie group, then the same construction with smooth instead of continuous functions leads to the corresponding *n-th smooth Čech cohomology.*

Remark 3.3.11. (cf. [De53, Section 12] and [GM99, 3.2.3]) If $n = 0, 1$, then we can perform a similar construction as in the previous remark in the case of a not necessarily abelian group K. The definition of an *n-cochain* is the same as in the commutative case, but we run into problems when writing down the boundary operator ∂. However, we may define $\partial_0(f)_{ij} = f_i \cdot f_j^{-1}$, $\partial_1(k)_{ijl} = k_{ij} \cdot k_{jl} \cdot k_{li}$ and call the elements of $\partial_1^{-1}(\{e\})$ *2-cocycles* (or *cocycles*, for short).

The way to circumvent difficulties for $n = 1$ is the observation that even in the non-abelian case, $C^1_c(\mathcal{U}, K)$ acts on cocycles by $f_i.k_{ij} = f_i \cdot k_{ij} \cdot f_j^{-1}$. Thus we define two cocycles k_{ij} and k'_{ij} to be equivalent if $k'_{ij} = f_i \cdot k_{ij} \cdot f_j^{-1}$ on U_{ij} for some $f_i \in C^1(\mathcal{U}, K)$, and by $\check{H}^1_c(\mathcal{U}, K)$ the equivalence classes (or the orbit space) of this action. Then $\check{H}^1_c(\mathcal{U}, K)$ is not a group, but we may nevertheless take the direct limit

$$\check{H}^1_c(M, K) := \lim_{\to} \check{H}^1_c(\mathcal{U}, K)$$

of sets and define it to be the 1^{st} *(non-abelian) continuous Čech cohomology* of M with coefficients in K. By its construction, $\check{H}^1_c(M, K)$ can also be viewed as the set of equivalence classes of continuous principal K-bundles over M (cf. Remark B.1.9).

Again, if M is a smooth manifold with corners and K is a Lie group, we can adopt this construction to define the 1^{st} *(non-abelian) smooth Čech cohomology* $\check{H}^1_s(M, K)$. ■

Theorem 3.3.12. *If M is a finite-dimensional paracompact manifold with corners and K is a Lie group modelled on a locally convex space, then the canonical map*

$$\iota : \check{H}^1_s(M, K) \to \check{H}^1_c(M, K)$$

is a bijection.

Proof. We identify smooth and continuous principal bundles with Čech 1-cocycles and smooth and continuous bundle equivalences with Čech 0-cochains as in Remark B.1.9. For each open cover $\mathcal{U}$ of M, we have the canonical map $\check{H}^1_s(\mathcal{U},K) \to \check{H}^1_c(\mathcal{U},K)$. Now each cocycle $k_{ij} : U_{ij} \to K$ defines a principal K-bundle $\mathcal{P}$ with locally trivial cover $\mathcal{U}$. We may assume by Theorem 3.3.5 that $\mathcal{P}$ is continuously equivalent to a smooth principal bundle $\widetilde{\mathcal{P}}$, and thus that $\mathcal{U}$ is also a locally trivial covering for $\widetilde{\mathcal{P}}$. This shows that the map is surjective, and the injectivity follows from Theorem 3.3.6 in the same way. Accordingly, the map induced on the direct limit is a bijection. ■

As a special case, we now consider principal bundles, whose structure groups is the projective unitary group of an infinite-dimensional Hilbert space $\mathcal{H}$.

Remark 3.3.13. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and denote by $\mathrm{U}(\mathcal{H})$ the group of unitary operators. If we equip $\mathrm{U}(\mathcal{H})$ with the norm topology, then the exponential series, restricted to skew-self-adjoint operators $\mathrm{L}(\mathrm{U}(\mathcal{H}))$, induces a Banach–Lie group structure on $\mathrm{U}(\mathcal{H})$ (cf. [Mi84, Ex. 1.1]). Then $\mathrm{U}(1) \cong Z(U(\mathcal{H}))$ and it can also be shown that $\mathrm{PU}(\mathcal{H}) := \mathrm{U}(\mathcal{H})/\,\mathrm{U}(1)$ is a Lie group modelled on $\mathrm{L}(\mathrm{U}(\mathcal{H}))/i\mathbb{R}$. ■

Remark 3.3.14. If X is a topological space with non-trivial n-th homotopy group $\pi_n(X)$ for all but one $n \in \mathbb{N}$, then it is called an *Eilenberg–MacLane space* $K(n, \pi_n(X))$. Since $\mathrm{U}(1)$ is a $K(1,\mathbb{Z})$, the long exact homotopy sequence [Br93, Theorem VII.6.7] shows that $\mathrm{PU}(\mathcal{H})$ is a $K(2,\mathbb{Z})$, since $\mathrm{U}(\mathcal{H})$ is contractible [Ku65, Theorem 3]. By the same argument, the classifying space $B\ \mathrm{PU}(\mathcal{H})$ is a $K(3,\mathbb{Z})$, since its total space $E\ \mathrm{PU}(\mathcal{H})$ is contractible (cf. Corollary B.2.7). Thus

$$\check{H}^3(M,\mathbb{Z}) \cong [M, B\ \mathrm{PU}(\mathcal{H})] \cong \check{H}^1_c(M, \mathrm{PU}(\mathcal{H}))$$

by [Br93, Corollary VII.13.16]. The representing class $[\mathcal{P}]$ in $\check{H}^3(M,\mathbb{Z})$ is called the *Dixmier–Douady class* of $\mathcal{P}$ (cf. [CCM98], [DD63]). It describes the obstruction of $\mathcal{P}$ to be the projectivisation of an (automatically trivial) principal $\mathrm{U}(\mathcal{H})$-bundle. ■

Corollary 3.3.15. *If M is a paracompact manifold with corners, then*

$$\check{H}^3(M,\mathbb{Z}) \cong \check{H}^1_c(M, \mathrm{PU}(\mathcal{H})) \cong \check{H}^1_s(M, \mathrm{PU}(\mathcal{H})).$$ ■

Bundles with $\mathrm{PU}(\mathcal{H})$ as structure group have an interesting application, because they are the key-ingredient for twisted K-theory.

Example 3.3.16 (Twisted K-theory). (cf. [Ro89, Secttion 2], [BCM$^+$02]) The Dixmier-Douady class of a principal $\mathrm{PU}(\mathcal{H})$-bundle over M induces a twisting of the K-theory of M in the following manner. For any paracompact space, the K-theory $K^0(M)$ is defined to be the Grothendieck group of the monoid of equivalence classes of finite-dimensional complex vector bundles over X, where addition and multiplication is defined by taking direct sums and tensor products of vector bundles [Hu94]. Furthermore, the space of Fredholm operators $\mathrm{Fred}(\mathcal{H})$ is a representing space for K-theory, i.e., $K^0(M) \cong [M, \mathrm{Fred}(\mathcal{H})]$, where $[\cdot,\cdot]$ denotes homotopy classes of continuous maps. Since $\mathrm{PU}(\mathcal{H})$ acts (continuously) on $\mathrm{Fred}(\mathcal{H})$ by conjugation, we can form the associated vector bundle $\mathcal{P}_{\mathrm{Fred}(\mathcal{H})} := \mathrm{Fred}(\mathcal{H}) \times_{\mathrm{PU}(\mathcal{H})} \mathcal{P}$. Then the homotopy classes of sections $[M, P_{\mathrm{Fred}(\mathcal{H})}]$ (or equivalently, the equivariant homotopy classes of equivariant maps $[P_{\mathrm{Fred}(\mathcal{H})}, \mathrm{Fred}(\mathcal{H})]^{\mathrm{PU}(\mathcal{H})}$) define the *twisted K-theory* $K_{\mathcal{P}}(M)$. Now Theorem 3.3.5 implies that we may assume $\mathcal{P}$ to be smooth. Since the action of $\mathrm{PU}(\mathcal{H})$ on $\mathrm{Fred}(\mathcal{H})$ is locally given by conjugation, it is smooth, whence is $\mathcal{P}_{\mathrm{Fred}(\mathcal{H})}$. Due to Lemma 3.3.7, we may, in the computation of $K_{\mathcal{P}}(M)$, restrict our attention to smooth sections and smooth homotopies. ■

3.4 The automorphism group as an infinite-dimensional Lie group

In this section we describe the Lie group structure on $\mathrm{Aut}(\mathcal{P})$ for a principal K-bundle over a compact manifold M *without* boundary, i.e., a *closed* compact manifold. We will do this using the extension of abstract groups

$$\mathrm{Gau}(\mathcal{P}) \hookrightarrow \mathrm{Aut}(\mathcal{P}) \xrightarrow{Q} \mathrm{Diff}(M)_{\mathcal{P}}, \tag{3.11}$$

where $\mathrm{Diff}(M)_{\mathcal{P}}$ is the image of the homomorphism $Q : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Diff}(M)$, $F \mapsto F_M$ from Definition 3.1.1. We will use this extension of abstract groups to construct a Lie group structure on $\mathrm{Aut}(\mathcal{P})$, being induced from the Lie group structures on $\mathrm{Gau}(\mathcal{P})$ from Section 3.1 and the classical one on $\mathrm{Diff}(M)$ (cf. [Le67], [Mi80], [KM97, Theorem 43.1] and [Gl06]). More precisely, we will construct a Lie group structure on $\mathrm{Aut}(\mathcal{P})$ that turns (3.11) into an extension of Lie groups, i.e., into a locally trivial bundle.

We shall consider bundles over bases without boundary, i.e., our base manifolds will always be closed compact manifolds. Throughout this section we fix one particular given principal K-bundle $\mathcal{P}$ over a closed compact manifold M and we furthermore assume that $\mathcal{P}$ has the property SUB.

We first clarify what we are aiming for.

Definition 3.4.1 (Extension of Lie groups). If N, $\widehat{G}$ and G are Lie groups, then an extension of groups

$$N \hookrightarrow \widehat{G} \twoheadrightarrow G$$

is called an *extension of Lie groups* if N is a split Lie subgroup of $\widehat{G}$. That means that $(N, q : \widehat{G} \to G)$ is a smooth principal N-bundle, where $q : \widehat{G} \to G \cong \widehat{G}/N$ is the induced quotient map. We call two extensions $N \hookrightarrow \widehat{G}_1 \twoheadrightarrow G$ and

$N \hookrightarrow \widehat{G}_2 \twoheadrightarrow G$ *equivalent* if there exists a morphism of Lie groups $\psi : \widehat{G}_1 \to \widehat{G}_2$ such that the diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d, "\mathrm{id}_N"'] & \widehat{G}_1 \arrow[r] \arrow[d, "\psi"'] & G \arrow[d, "\mathrm{id}_G"'] \\ N \arrow[r] & \widehat{G}_2 \arrow[r] & G \end{tikzcd}$$

commutes. ■

These extensions are treated in detail in [Ne06a], where it is shown that they are parametrised by smooth local data arising from smooth local sections $s : O \to \widehat{G}$ of q, where $O \subseteq G$ is an open unit neighbourhood. We will not use the whole framework from [Ne06a] rather than using the idea that we need to construct a section of Q on some unit neighbourhood of $\mathrm{Diff}(M)$ that has certain smoothness properties.

Throughout this section we have to work with trivialising systems that have some nice properties in order to make the constructions work. This we collect in the following remark.

Remark 3.4.2. Unless stated otherwise, for the rest of this section we choose and fix one particular smooth closed trivialising system $\overline{\mathcal{V}} = (\overline{V}_i, \sigma_i)_{i=1,\dots,n}$ of $\mathcal{P}$ such that

- each $\overline{V}_i$ is a compact manifold with corners diffeomorphic to $[0,1]^{\dim(M)}$,
- $\overline{\mathcal{V}}$ is a refinement of some smooth open trivialising system $\mathcal{U} = (U_i, \tau_i)_{i=1,\dots,n}$ and we have $\overline{V}_i \subseteq U_i$ and $\sigma_i = \tau_i|_{\overline{V}_i}$,
- each $\overline{U}_i$ is a compact manifold with corners diffeomorphic to $[0,1]^{\dim(M)}$ and τ_i extends to a smooth section $\tau_i : \overline{U}_i \to P$,
- $\overline{\mathcal{U}} = (\overline{U}_i, \tau_i)_{i=1,\dots,n}$ is a refinement of some smooth open trivialising system $\mathcal{U}' = (U_i', \tau_j)_{j=1,\dots,m}$,
- the values of the transition functions $k_{ij} : U_i' \cap U_j' \to K$ of $\mathcal{U}'$ are contained in open subsets W_{ij} of K, which are diffeomorphic to open zero neighbourhoods of $\mathfrak{k}$,
- $\mathcal{P}$ has the property SUB with respect to $\overline{\mathcal{V}}$ (and thus with respect to $\overline{\mathcal{U}}$ by Lemma 3.1.12).

We choose $\overline{\mathcal{V}}$ by starting with an arbitrary smooth closed trivialising system such that $\mathcal{P}$ has the property SUB with respect to this system. Note that this exists because we assume throughout this section that $\mathcal{P}$ has the property SUB. Then Lemma B.1.8 implies that there exists a refinement $\mathcal{U}' = (U_j', \tau_j)_{j=1,\dots,m}$ such that the transition functions $k_{ij} : U_i \cap U_j \to K$ take values in open subsets W_{ij} of K, which are diffeomorphic to open convex zero neighbourhoods of $\mathfrak{k}$. Now each $x \in M$ has neighbourhoods V_x and U_x such that $\overline{V}_x \subseteq U_x$, $\overline{V}_x$ and $\overline{U}_x$ are diffeomorphic to $[0,1]^{\dim(M)}$ and $\overline{U}_x \subseteq U_{j(x)}$ for some $j(x) \in \{1,\dots,m\}$. Then finitely many $V_{x_1},\dots,V_{x_n}$ cover M and so do $U_{x_1},\dots,U_{x_n}$. Furthermore, the sections τ_j restrict to smooth sections on V_i, $\overline{V}_i$, U_i and $\overline{U}_i$.

This choice of $\overline{\mathcal{U}}$ in turn implies that $k_{ij}|_{\overline{U}_i \cap \overline{U}_j}$ arises as the restriction of some smooth function on M. In fact, if $\varphi_{ij} : W_{ij} \to W'_{ij} \subseteq \mathfrak{k}$ is a diffeomorphism onto a convex zero neighbourhood and $f_{ij} \in C^\infty(M,\mathbb{R})$ is a smooth function with $f_{ij}|_{\overline{U}_i \cap \overline{U}_j} \equiv 1$ and $\operatorname{supp}(f_{ij}) \subseteq U'_i \cap U'_j$, then

$$m \mapsto \begin{cases} \varphi_{ij}^{-1}(f_{ij}(m) \cdot \varphi_{ij}(k_{ij}(m))) & \text{if } m \in U'_i \cap U'_j \\ \varphi_{ij}^{-1}(0) & \text{if } m \notin U'_i \cap U'_j \end{cases}$$

is a smooth function, because each $m \in \partial(U'_i \cap U'_j)$ has a neighbourhood on which f_{ij} vanishes, and this function coincides with k_{ij} on $\overline{U}_i \cap \overline{U}_j$.

Similarly, let $(\gamma_1,\dots,\gamma_n) \in G_{\overline{\mathcal{U}}}(\mathcal{P}) \subseteq \prod_{i=1}^n C^\infty(\overline{U}_i, K)$ be the local description of some $\gamma \in C^\infty(P,K)^K$. We will show that each $\gamma_i|_{\overline{V}_i}$ arises as the restriction of a smooth map on M. In fact, take a diffeomorphism $\varphi_i : \overline{U}_i \to [0,1]^{\dim(M)}$. Then $\overline{V}_i \subseteq U_i$ implies $\varphi_i(\overline{V}_i) \subseteq (0,1)^{\dim(M)}$ and thus there exits an $\varepsilon > 0$ such that $\varphi_i(\overline{V}_i) \subseteq (\varepsilon, 1-\varepsilon)^{\dim(M)}$ for all $i = 1,\dots,n$. Now let

$$f : [0,1]^{\dim(M)} \backslash (\varepsilon, 1-\varepsilon)^{\dim(M)} \to [\varepsilon, 1-\varepsilon]^{\dim(M)}$$

be a map that restricts to the identity on $\partial[\varepsilon,1-\varepsilon]^{\dim(M)}$ and collapses $\partial[0,1]^{\dim(M)}$ to a single point x_0. We then set

$$\gamma'_i : M \to K \quad m \mapsto \begin{cases} \gamma_i(m) & \text{if } m \in \overline{U}_i, \varphi_i(m) \in [\varepsilon, 1-\varepsilon]^{\dim(M)} \\ \gamma_i(\varphi_i^{-1}(f(\varphi_i(m)))) & \text{if } m \in \overline{U}_i, \varphi_i(m) \notin (\varepsilon, 1-\varepsilon)^{\dim(M)} \\ \gamma_i(\varphi_i^{-1}(x_0)) & \text{if } m \notin U_i, \end{cases}$$

and γ'_i is well-defined and continuous, because $f(\varphi_i(m)) = \varphi_i(m)$ if $\varphi_i(m) \in \partial[\varepsilon,1-\varepsilon]^{\dim(M)}$ and $f(\varphi_i(m)) = x_0$ if $\varphi_i(m) \in \partial[0,1]^{\dim(M)}$. Since γ'_i coincides with γ_i on the neighbourhood $\varphi_i^{-1}((\varepsilon,1-\varepsilon)^{\dim(M)})$, it thus is smooth on this neighbourhood. Now Proposition 3.2.8, applied to the closed set $\overline{V}_i$ and the open set $M \backslash \overline{V}_i$ yields a smooth map $\widetilde{\gamma}_i$ on M with $\gamma_i|_{\overline{V}_i} = \widetilde{\gamma}_i|_{\overline{V}_i}$. ■

We now give the description of a strategy for lifting special diffeomorphisms to bundle automorphisms. This should motivate the procedure of this section.

Remark 3.4.3. Let $U \subseteq M$ be open and trivialising with section $\sigma : U \to P$ and corresponding $k_\sigma : \pi^{-1}(U) \to K$, given by $\sigma(\pi(p)) \cdot k_\sigma(p) = p$. If $g \in \operatorname{Diff}(M)$ is such that $\operatorname{supp}(g) \subseteq U$, then we may define a smooth bundle automorphism $\widetilde{g}$ by

$$\widetilde{g}(p) = \begin{cases} \sigma\left(g\left(\pi(p)\right)\right) \cdot k(p) & \text{if } p \in \pi^{-1}(U) \\ p & \text{else,} \end{cases}$$

because each $x \in \partial U$ has a neighbourhood on which g is the identity. Furthermore, one easily verifies $Q(\widetilde{g}) = \widetilde{g}_M = g$ and $\widetilde{g^{-1}} = \widetilde{g}^{-1}$, where $Q : \operatorname{Aut}(\mathcal{P}) \to \operatorname{Diff}(M)$ is the homomorphism from Definition B.1.4. ■

The procedure is now as follows. For a suitable identity neighbourhood $O \subseteq \operatorname{Diff}(M)$ we decompose $g \in O$ into $g_1, \ldots, g_n$ such that $\operatorname{supp}(g_i) \subseteq V_i$. Each g_i can be lifted by the preceding remark to $\widetilde{g_i} \in \operatorname{Aut}(\mathcal{P})$ and then $\widetilde{g_n} \circ \ldots \circ \widetilde{g_1}$ will be the lift of g to $\operatorname{Aut}(\mathcal{P})$. In order to perform the mentioned decomposition, we need to know some basics on the charts, turning $\operatorname{Diff}(M)$ into a Fréchet–Lie group modelled on the space of vector fields $\mathcal{V}(M)$.

Remark 3.4.4 (Charts for Diffeomorphism Groups). Let M be a closed compact manifold with a fixed Riemannian metric g and let $\pi : TM \to M$ be its tangent bundle and $\operatorname{Exp} : TM \to M$ be the exponential mapping of g. Then $\pi \times \operatorname{Exp} : TM \to M \times M,\ X_m \mapsto (m, \operatorname{Exp}(X_m))$ restricts to a diffeomorphism on an open neighbourhood U of the zero section in TM. We set $O' := \{X \in \mathcal{V}(M) : X(M) \subseteq U\}$ and define

$$\varphi^{-1} : O' \to C^\infty(M, M), \quad \varphi^{-1}(X)(m) = \operatorname{Exp}(X(m))$$

For the following, observe that $\varphi^{-1}(X)(m) = m$ if and only if $X(m) = 0_m$. After shrinking O' to a convex open neighbourhood in the C^1-topology, one can also ensure that $\varphi^{-1}(X) \in \operatorname{Diff}(M)$ for all $X \in O'$. Since $\pi \times \operatorname{Exp}$ is bijective on U, φ^{-1} maps O' bijectively to $O := \varphi^{-1}(O') \subseteq \operatorname{Diff}(M)$ and thus endows O with a smooth manifold structure. Furthermore, it can be shown that in view of Proposition A.1.6, this chart actually defines a Lie group structure on $\operatorname{Diff}(M)$ (cf. [Le67], [KM97, Theorem 43.1] or [Gl06]). It is even possible to put Lie group structures on $\operatorname{Diff}(M)$ in the case of non-compact manifolds, possibly with corners [Mi80, Theorem 11.11], but we will not go into this generality here. ■

Lemma 3.4.5. *For the open cover $V_1, \ldots, V_n$ of the closed compact manifold M and the open identity neighbourhood $O \subseteq \operatorname{Diff}(M)$ from Remark 3.4.4, there exist smooth maps*

$$s_i : O \to O \circ O^{-1} \tag{3.12}$$

for $1 \leq i \leq n$ such that $\operatorname{supp}(s_i(g)) \subseteq V_i$ and $s_n(g) \circ \ldots \circ s_1(g) = g$.

Proof. (cf. [HT04, Proposition 1]) Let $f_1, \ldots, f_n$ be a partition of unity subordinated to the open cover $V_1, \ldots, V_n$ and let $\varphi : O \to \varphi(O) \subseteq \mathcal{V}(M)$ be the chart of $\operatorname{Diff}(M)$ form Remark 3.4.4. In particular, $\varphi^{-1}(X)(m) = m$ if $X(m) = 0_m$. Since $\varphi(O)$ is convex, we may define $s_i : O \to O \circ O^{-1}$,

$$s_i(g) = \varphi^{-1}\big((f_n + \ldots + f_i) \cdot \varphi(g)\big) \circ \big(\varphi^{-1}\big((f_n + \ldots + f_{i+1}) \cdot \varphi(g)\big)\big)^{-1}$$

if $i < n$ and $s_n(g) = \varphi^{-1}(f_n \cdot \varphi(g))$, which are smooth since they are given by a push-forward of the smooth map $\mathbb{R} \times TM \to TM\ \ (\lambda, X_m) \mapsto \lambda \cdot X_m$. Furthermore, if $f_i(x) = 0$, then the left and the right factor annihilate each other and thus $\operatorname{supp}(s_i(g)) \subseteq V_i$. ■

As mentioned above, the preceding lemma enables us now to lift elements of $O \subseteq \mathrm{Diff}(M)$ to elements of $\mathrm{Aut}(\mathcal{P})$.

Definition 3.4.6. If $O \subseteq \mathrm{Diff}(M)$ is the open identity neighbourhood from Remark 3.4.4 and $s_i : O \to O \circ O^{-1}$ are the smooth mappings from Lemma 3.4.5, then we define

$$S : O \to \mathrm{Aut}(\mathcal{P}), \quad g \mapsto S(g) := \widetilde{g_n} \circ \ldots \circ \widetilde{g_1}, \tag{3.13}$$

where $\widetilde{g_i}$ is the bundle automorphism of $\mathcal{P}$ from Remark 3.4.3. This defines a local section for the homomorphism $Q : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Diff}(M)$, $F \mapsto F_M$ from Definition 3.1.1. ■

We shall frequently need an explicit description of $S(g)$ in terms of local trivialisations, i.e., how $S(g)(\sigma_i(x))$ can be expressed in terms of g_j, σ_j and $k_{jj'}$.

Remark 3.4.7. Let $x \in V_i \subseteq M$ be such that $x \notin V_j$ for $j < i$ and $g_i(x) \notin V_j$ for $j > i$. Then $g_j(x) = x$ for all $j < i$, $g_j(g_i(x)) = g_i(x)$ for all $j > i$ and thus $S(g)(\sigma_i(x)) = \sigma_i(g_i(x)) = \sigma_i(g(x))$.

In general, things are more complicated. The first $\widetilde{g_{j_1}}$ in (3.13) that could move $\sigma_i(x)$ is the one for the minimal j_1 such that $x \in \overline{V}_{j_1}$. We then have

$$\widetilde{g_{j_1}}(\sigma_i(x)) = \widetilde{g_{j_1}}(\sigma_{j_1}(x)) \cdot k_{j_1 i}(x) = \sigma_{j_1}(g_{j_1}(x)) \cdot k_{j_1 i}(x).$$

The next $\widetilde{g_{j_2}}$ in (3.13) that could move $\widetilde{g_{j_1}}(\sigma_i(x))$ in turn is the one for the minimal $j_2 > j_1$ such that $g_{j_1}(x) \in \overline{V}_{j_2}$, and we then have

$$\widetilde{g_{j_2}}(\widetilde{g_{j_1}}(\sigma_i(x))) = \sigma_{j_2}(g_{j_2} \circ g_{j_1}(x)) \cdot k_{j_2 j_1}(g_{j_1}(x)) \cdot k_{j_1 i}(x).$$

We eventually get

$$S(g)(\sigma_i(x)) = \sigma_{j_\ell}(g(x)) \cdot k_{j_\ell j_{\ell-1}}(g_{j_{\ell-1}} \circ \ldots \circ g_{j_1}(x)) \cdot \ldots \cdot k_{j_1 i}(x), \tag{3.14}$$

where $\{j_1, \ldots, j_\ell\} \subseteq \{1, \ldots, n\}$ is maximal such that

$$g_{j_{p-1}} \circ \ldots \circ g_{i_1}(x) \in U_{j_p} \cap U_{j_{p-1}} \text{ for } 2 \leq p \leq \ell \text{ and } j_1 < \ldots < j_p.$$

Note that we cannot write down such a formula using all $j \in \{1, \ldots, n\}$, because the corresponding $k_{jj'}$ and σ_j would not be defined properly.

Of course, g and x influence the choice of $j_1, \ldots, j_\ell$, but there exist open neighbourhoods O_g of g and U_x of x such that we may use (3.14) as a formula for all $g' \in O_g$ and $x' \in U_x$. In fact, the action $\mathrm{Diff}(M) \times M \to M$, $g.m = g(m)$ is smooth ([Gl06, Proposition 7.2]), and thus in particular continuous. If

$$g_{j_p} \circ \ldots \circ g_{j_1}(x) \notin \overline{V}_j \text{ for } 2 \leq p \leq \ell \text{ and } j \notin \{j_1, \ldots, j_p\} \tag{3.15}$$

$$g_{j_p} \circ \ldots \circ g_{j_1}(x) \in U_{j_p} \cap U_{j_{p-1}} \text{ for } 2 \leq p \leq \ell \text{ and } j_1 < \ldots < j_p \tag{3.16}$$

then this is also true for g' and x' in some open neighbourhood of g and x. This yields finitely many open open neighbourhoods of g and x and we define their intersections to be O_g and U_x. Then (3.14) still holds for $g' \in O_g$ and $x' \in U_x$, because (3.15) implies $g_j(g_{j_p} \circ \ldots \circ g_{j_1}(x)) = g_{j_p} \circ \ldots \circ g_{j_1}(x)$ and (3.16) implies that $k_{j_p j_{p-1}}$ is defined and satisfies the cocycle condition. ■

In order to determine a Lie group structure on $\operatorname{Aut}(\mathcal{P})$, the map $S : O \to \operatorname{Aut}(\mathcal{P})$ has to satisfy certain smoothness properties. To motivate this, assume that $\operatorname{Aut}(\mathcal{P})$ already has a smooth structure and that $S : O \to \operatorname{Aut}(\mathcal{P})$ is smooth. Then the two maps

$$\begin{aligned} T : \operatorname{Gau}(\mathcal{P}) \times O \to \operatorname{Aut}(\mathcal{P}), &\quad (F, g) \mapsto S(g) \circ F \circ S(g)^{-1} \\ \omega : O \times O \to \operatorname{Aut}(\mathcal{P}), &\quad (g, g') \mapsto S(g) \circ S(g') \circ S(g \circ g')^{-1} \end{aligned}$$

are also smooth. Moreover, for each $g \in \operatorname{Diff}(M)_{\mathcal{P}}$, there exists an open identity neighbourhood $O_g \subseteq O$ such that $g \circ O_g \circ g^{-1} \subseteq O$ and that

$$\omega_g : O_g \to \operatorname{Aut}(\mathcal{P}), \quad g' \mapsto F \circ S(g') \circ F^{-1} \circ S(g \circ g' \circ g^{-1})^{-1}$$

is smooth for any $F \in \operatorname{Aut}(\mathcal{P})$ with $F_M = g$.

Now T, ω and ω_g actually take values in $\operatorname{Gau}(\mathcal{P}) = \ker(Q)$, because $Q : \operatorname{Aut}(\mathcal{P}) \to \operatorname{Diff}(M)_{\mathcal{P}}$ is a homomorphism and $Q \circ S = \operatorname{id}_O$. It thus makes sense to require these maps to be smooth, even if we do not jet have a smooth structure on $\operatorname{Aut}(\mathcal{P})$. However, we will see later that requiring these maps to be smooth determines a smooth structure on $\operatorname{Aut}(\mathcal{P})$. More generally speaking, (T, ω) is (the restriction of) a *smooth factor system* or *locally smooth 2-cocycle* for $(\operatorname{Gau}(\mathcal{P}), \operatorname{Diff}(M)_{\mathcal{P}})$ in the sense of [Ne06a]. These smooth factor systems parametrise the set of non-abelian extensions of $\operatorname{Diff}(M)_{\mathcal{P}}$ by $\operatorname{Gau}(\mathcal{P})$ [Ne06a, Proposition II.8].

Since we can access the smooth structure on $\operatorname{Gau}(\mathcal{P})$ only via the isomorphism $\operatorname{Gau}(\mathcal{P}) \cong C^\infty(P, K)^K$ we first relate the conjugation action of $\operatorname{Aut}(\mathcal{P})$ on $\operatorname{Gau}(\mathcal{P})$ to the corresponding action of $\operatorname{Aut}(\mathcal{P})$ on $C^\infty(P, K)^K$.

Remark 3.4.8. If we identify the normal subgroup $\operatorname{Gau}(\mathcal{P}) \trianglelefteq \operatorname{Aut}(\mathcal{P})$ with $C^\infty(P, K)^K$ via

$$C^\infty(P, K)^K \to \operatorname{Gau}(\mathcal{P}), \quad \gamma \mapsto F_\gamma$$

with $F_\gamma(p) = p \cdot \gamma(p)$, then the conjugation action $\operatorname{Aut}(\mathcal{P}) \times \operatorname{Gau}(\mathcal{P}) \to \operatorname{Gau}(\mathcal{P})$, given by $(F, F_\gamma) \mapsto F \circ F_\gamma \circ F^{-1}$ changes into

$$c : \operatorname{Aut}(\mathcal{P}) \times C^\infty(P, K)^K \to C^\infty(P, K)^K, \quad (F, \gamma) \mapsto \gamma \circ F^{-1}.$$

In fact, this follows from

$$(F \circ F_\gamma \circ F^{-1})(p) = F\big(F^{-1}(p) \cdot \gamma(F^{-1}(p))\big) = p \cdot \gamma(F^{-1}(p)) = F_{(\gamma \circ F^{-1})}(p). \quad ■$$

In the following remarks and lemmas we show the smoothness of the maps T, ω and ω_g, mentioned before.

Lemma 3.4.9. *Let $O \subseteq \operatorname{Diff}(M)$ be the open identity neighbourhood from Remark 3.4.4 and $S : O \to \operatorname{Aut}(\mathcal{P})$ be the map from Definition 3.4.6. For each $F \in \operatorname{Aut}(\mathcal{P})$ the map $C^\infty(P,\mathfrak{k})^K \to C^\infty(P,\mathfrak{k})^K$, $\eta \mapsto \eta \circ F^{-1}$ is an automorphism of $C^\infty(P,\mathfrak{k})^K$ and the map*

$$t : C^\infty(P,\mathfrak{k})^K \times O \to C^\infty(P,\mathfrak{k})^K, \quad (\eta, g) \mapsto \eta \circ S(g)^{-1}$$

is smooth.

Proof. That $\eta \mapsto \eta \circ F^{-1}$ is an element of $\operatorname{Aut}(C^\infty(P,\mathfrak{k})^K)$ follows immediately from the (pointwise) definition of the bracket on $C^\infty(P,\mathfrak{k})^K$ and Lemma 2.2.24. We shall use the isomorphism $C^\infty(P,\mathfrak{k})^K \cong \mathfrak{g}_{\mathcal{U}'}(\mathcal{P}) \cong \mathfrak{g}_{\mathcal{U}}(\mathcal{P}) \cong \mathfrak{g}_{\mathcal{V}}(\mathcal{P})$ from Proposition 3.1.4 and reduce the smoothness of t to the smoothness of

$$C^\infty(M,\mathfrak{k}) \times \operatorname{Diff}(M) \to C^\infty(M,\mathfrak{k}), \quad (\eta, g) \mapsto \eta \circ g^{-1}$$

from Lemma 2.2.25 and to the action of g_i^{-1} on $C^\infty(\overline{V}_i,\mathfrak{k})$, because we have no description of what g_i^{-1} does with U_j for $j \neq i$. It clearly suffices to show that the map

$$t_i : C^\infty(P,\mathfrak{k})^K \times \operatorname{Diff}(M) \to C^\infty(P,\mathfrak{k})^K \times \operatorname{Diff}(M), \quad (\eta, g) \mapsto (\eta \circ \widetilde{g_i}^{-1}, g)$$

is smooth for each $1 \leq i \leq n$, because then $t = \mathrm{pr}_1 \circ t_n \circ \ldots \circ t_1$ is smooth. This in turn follows from the smoothness of

$$C^\infty(U_i',\mathfrak{k}) \times \operatorname{Diff}(M) \to C^\infty(U_i,\mathfrak{k}), \quad (\eta, g) \mapsto \eta \circ g_i^{-1}\big|_{U_i}, \tag{3.17}$$

because this is the local description of t_i. In fact, for each $j \neq i$ there exists an open subset V_j' with $U_j \backslash U_i \subseteq V_j' \subseteq U_j \backslash V_i$, because $\overline{V}_i \subseteq U_i$ and U_j is diffeomorphic to $(0,1)^{\dim(M)}$. Furthermore, we set $V_i' := U_i$. Then $(V_1', \ldots, V_n')$ is an open cover of M, leading to a refinement $\mathcal{V}'$ of the trivialising system $\mathcal{U}'$ and we have

$$t_i : \mathfrak{g}_{\mathcal{U}'}(\mathcal{P}) \times O \to \mathfrak{g}_{\mathcal{V}'}(\mathcal{P}), \quad ((\eta_1, \ldots, \eta_n), g) \mapsto (\eta_1|_{V_1'}, \ldots, \eta_i \circ g_i^{-1}\big|_{V_i'}, \ldots, \eta_n|_{V_n'})$$

because $\operatorname{supp}(g_i) \subseteq V_i$ and $V_j' \cap V_i = \emptyset$ if $j \neq i$. To show that (3.17) is smooth, choose some $f_i \in C^\infty(M,\mathbb{R})$ with $f_i|_{U_i} \equiv 1$ and $\operatorname{supp}(f_i) \subseteq U_i'$. Then

$$h_i : C^\infty(U_i',\mathfrak{k}) \to C^\infty(M,\mathfrak{k}), \quad \eta \mapsto \left(m \mapsto \begin{cases} f_i(m) \cdot \eta(m) & \text{if } m \in U_i' \\ 0 & \text{if } m \notin U_i' \end{cases} \right)$$

is smooth by Corollary 2.2.10, because $\eta \mapsto f_i|_{U_i'} \cdot \eta$ is linear, continuous and thus smooth. Now we have $\operatorname{supp}(g_i) \subseteq V_i \subseteq U_i$ and thus $h_i(\eta) \circ g_i^{-1}\big|_{U_i} = \eta \circ g_i^{-1}\big|_{U_i}$ depends smoothly on g and η by Corollary 2.2.8. ■

The following proofs share a common idea. We will always have to show that certain mappings with values in $C^\infty(P,K)^K$ are smooth. This can be established by showing that their compositions with the pull-back $(\sigma_i)^*$ of a section $\sigma_i : \overline{V}_i \to P$ (then with values in $C^\infty(\overline{V}_i, K)$) are smooth for all $1 \leq i \leq n$.

As described in Remark 3.4.7, it will not be possible to write down explicit formulas for these mappings in terms of the transition functions k_{ij} for all $x \in \overline{V}_i$ simultaneously, but we will be able to do so on some open neighbourhood U_x of x. For different x_1 and x_2 these formulas will define the same mapping on $U_{x_1} \cap U_{x_2}$, because there they define $(\sigma_i^*(S(g))) = S(g) \circ \sigma_i$. By restriction and gluing we will thus be able to reconstruct the original mappings and then see that they depend smoothly on their arguments.

Lemma 3.4.10. *If $O \subseteq \mathrm{Diff}(M)$ is the open identity neighbourhood from Remark 3.4.4 and $S : O \to \mathrm{Aut}(\mathcal{P})$ is the map from Definition 3.4.6, then for each $\gamma \in C^\infty(P,K)^K$ the map*

$$O \ni g \mapsto \gamma \circ S(g)^{-1} \in C^\infty(P,K)^K$$

is smooth.

Proof. It suffices to show that $\gamma \circ S(g)^{-1} \circ \sigma_i|_{\overline{V}_i}$ depends smoothly on g for $1 \leq i \leq n$. Let $(\gamma_1, \ldots, \gamma_n) \in G_{\mathcal{U}}(\mathcal{P}) \subseteq \prod_{i=1}^n C^\infty(\overline{U}_i, K)$ be the local description of γ. Fix $g \in O$ and $x \in \overline{V}_i$. Then Remark 3.4.7 yields open neighbourhoods O_g of g and U_x of x (w.l.o.g. such that $\overline{U}_x \subseteq \overline{V}_i$ is a manifold with corners) such that

$$\begin{aligned}\gamma(S(g')^{-1}(\sigma_i(x'))) &= \gamma\big(\sigma_{j_\ell}(g'(x'))\underbrace{\cdot k_{j_\ell j_{\ell-1}}(g'_{j_{\ell-1}} \circ \ldots \circ g'_{j_1}(x')) \cdot \ldots \cdot k_{j_1 i}(x')}_{:=\kappa_{x,g'}(x')}\big) \\ &= \kappa_{x,g'}(x')^{-1} \cdot \gamma\big(\sigma_{j_\ell}(g'(x'))\big) \cdot \kappa_{x,g'}(x') = \underbrace{\kappa_{x,g'}(x')^{-1} \cdot \gamma_{j_\ell}(g'(x')) \cdot \kappa_{x,g'}(x')}_{:=\theta_{x,g'}(x')}\end{aligned}$$

for all $g' \in O_g$ and $x' \in \overline{U}_x$. Since we will not vary i and g in the sequel, we suppressed the dependence of $\kappa_{x,g'}(x')$ and $\theta_{x,g'}(x')$ on i and g. Note that each $k_{jj'}$ and γ_i can be assumed to be defined on M (cf. Remark 3.4.2). Thus, for fixed x, the formula for $\theta_{x,g'}$ defines a smooth function on M that depends smoothly on g', because the action of $\mathrm{Diff}(M)$ on $C^\infty(M,K)$ is smooth (cf. Proposition 2.2.27).

Furthermore, $\theta_{x_1,g'}$ and $\theta_{x_2,g'}$ coincide on $\overline{U}_{x_1} \cap \overline{U}_{x_2}$, because there they both define $\gamma \circ S(g')^{-1} \circ \sigma_i$. Now finitely many $U_{x_1}, \ldots, U_{x_m}$ cover $\overline{V}_i$, and since the gluing and restriction maps from Lemma 2.2.20 and Proposition 2.2.21 are smooth we have that

$$\gamma \circ S(g')^{-1} \circ \sigma_i = \mathrm{glue}(\theta_{x_1,g'}|_{\overline{U}_{x_1}}, \ldots, \theta_{x_m,g'}|_{\overline{U}_{x_m}})$$

depends smoothly on g'. ∎

The following two lemmas provide a smooth factor system (T,ω) for $(\mathrm{Gau}(\mathcal{P}),\mathrm{Diff}(M)_{\mathcal{P}})$.

Lemma 3.4.11. *Let $O\subseteq \mathrm{Diff}(M)$ be the open identity neighbourhood from Remark 3.4.4 and $S:O\to \mathrm{Aut}(\mathcal{P})$ be the map from Definition 3.4.6. For each $F\in \mathrm{Aut}(\mathcal{P})$ the map $c_F: C^{\infty}(P,K)^K\to C^{\infty}(P,K)^K$, $\gamma\mapsto\gamma\circ F^{-1}$ is an automorphism of $C^{\infty}(P,K)^K$ and the map*

$$T: C^{\infty}(P,K)^K\times O\to C^{\infty}(P,K)^K,\quad (\gamma,g)\mapsto \gamma\circ S(g)^{-1} \tag{3.18}$$

is smooth.

Proof. Since $\gamma\mapsto\gamma\circ F^{-1}$ is a group homomorphism, it suffices to show that it is smooth on a unit neighbourhood (Lemma A.3.3). Because the charts on $C^{\infty}(P,K)^K$ are constructed by push-forwards (cf. Proposition 3.1.8) this follows immediately from the fact that the corresponding automorphism of $C^{\infty}(P,\mathfrak{k})^K$, given by $\eta\mapsto\eta\circ F^{-1}$, is continuous and thus smooth. For the same reason, Lemma 3.4.9 implies that there exists a unit neighbourhood $U\subseteq C^{\infty}(P,K)^K$ such that

$$U\times O\to C^{\infty}(P,K)^K,\quad (\gamma,g)\mapsto\gamma\circ S(g)^{-1}$$

is smooth.

Now for each $\gamma_0\in C^{\infty}(P,K)^K$ there exists an open neighbourhood U_{γ_0} such that $\gamma_0^{-1}\cdot U_{\gamma_0}\subseteq U$. Hence

$$\gamma\circ S(g)^{-1}=(\gamma_0\cdot\gamma_0^{-1}\cdot\gamma)\circ S(g)^{-1}=\big(\gamma_0\circ S(g)^{-1}\big)\cdot\big((\gamma_0^{-1}\cdot\gamma)\circ S(g)^{-1}\big),$$

and the first factor depends smoothly on g due to Lemma 3.4.10, and the second factor depends smoothly on γ and g, because $\gamma_0^{-1}\cdot\gamma\in U$. ■

Lemma 3.4.12. *If $O\subseteq\mathrm{Diff}(M)$ is the open identity neighbourhood from Remark 3.4.4 and $S:O\to\mathrm{Aut}(\mathcal{P})$ is the map from Definition 3.4.6, then*

$$\omega: O\times O\to C^{\infty}(P,K)^K,\quad (g,g')\mapsto S(g)\circ S(g')\circ S(g\circ g')^{-1} \tag{3.19}$$

is smooth. Furthermore, if $Q:\mathrm{Aut}(\mathcal{P})\to\mathrm{Diff}(M)$, $F\mapsto F_M$ is the homomorphism from Definition 3.1.1 then for each $g\in Q(\mathrm{Diff}(M))$ there exists an open identity neighbourhood $O_g\subseteq O$ such that

$$\omega_g: O_g\to C^{\infty}(P,K)^K,\quad g'\mapsto F\circ S(g')\circ F^{-1}\circ S(g\circ g'\circ g^{-1})^{-1} \tag{3.20}$$

is smooth for any $F\in\mathrm{Aut}(\mathcal{P})$ with $F_M=g$.

Proof. First observe that $\omega(g,g')$ actually is an element of $C^{\infty}(P,K)^K\cong\mathrm{Gau}(\mathcal{P})=\ker(Q)$, because Q is a homomorphism of groups, S is a section of Q and thus

$$Q(\omega(g,g'))=Q(S(g))\circ Q(S(g'))\circ Q(S(g\circ g'))^{-1}=\mathrm{id}_M\,.$$

To show that ω is smooth, we derive an explicit formula for $\omega(g,g')\circ\sigma_i \in C^\infty(\overline{V}_i,K)$ that depends smoothly on g and g'.

Denote $\widehat{g} := g\circ g'$ for $g,g'\in O$ and fix $g,g'\in O$, $x\in\overline{V}_i$. Proceeding as in Remark 3.4.7, we find $i_1,\ldots,i_\ell$ such that

$$S(\widehat{g})^{-1}(\sigma_{i_\ell}(x)) = \sigma_\ell(\widehat{g}^{-1}(x))\cdot k_{i_\ell i_{\ell-1}}((\widehat{g}_{i_{\ell-1}})^{-1}\circ\ldots\circ(\widehat{g}_{i_1})^{-1}(x))\cdot\ldots\cdot k_{i_1 i}(x).$$

Accordingly we find $i'_{\ell'},\ldots,i'_1$ for $S(g')$ and $i''_{\ell''},\ldots,i''_1$ for $S(g)$. We get as in Remark 3.4.7 open neighbourhoods $O_g,O_{g'}$ of g,g' and U_x of x (w.l.o.g. such that $\overline{U}_x\subseteq\overline{V}_i$ is a manifold with corners) such that for $h\in O_g$, $h\in O_{g'}$ and $x'\in\overline{U}_x$ we have $S(h)\cdot S(h')\cdot S(h\cdot h')^{-1}(\sigma_i(x'))=$

$$\begin{aligned}\sigma_i(x')\cdot\Big[&k_{i\,i''_{\ell''}}(x')\\ &\cdot k_{i''_{\ell''}i''_{\ell''-1}}\big(h_{i''_{\ell''-1}}\circ\ldots\circ h_{i''_1}\circ h^{-1}(x')\big)\cdot\ldots\cdot k_{i''_1 i'_{\ell'}}(h^{-1}(x'))\\ &\cdot k_{i'_{\ell'}i'_{\ell'-1}}\big(h'_{i'_{\ell'-1}}\circ\ldots\circ h'_{i'_1}\circ\widehat{h}^{-1}(x')\big)\cdot\ldots\quad\cdot k_{i'_1 i_\ell}(\widehat{h}^{-1}(x'))\\ &\cdot k_{i_\ell i_{\ell-1}}\big((\widehat{h}_{i_{\ell-1}})^{-1}\circ\ldots\circ(\widehat{h}_{i_1})^{-1}(x')\big)\cdot\ldots\cdot k_{i_1 i}(x')\Big].\end{aligned}$$

Denote by $\kappa_{x,h,h'}(x')\in K$ the element in brackets on the right hand side, and note that it defines $\omega(h,h')\circ\sigma_i(x')$ by Remark 3.1.2. Since we will not vary g and g' in the sequel we suppressed the dependence of $\kappa_{x,h,h'}(x')$ on them.

Now each k_{ij} can be assumed to be defined on M (cf. Remark 3.4.2). Thus, for fixed x, the formula for $\kappa_{x,h,h'}$ defines a smooth function on M that depends smoothly on h and h', because the action of $\operatorname{Diff}(M)$ on $C^\infty(M,K)$ is smooth (cf. Proposition 2.2.27). Furthermore, $\kappa_{x_1,h,h'}$ coincides with $\kappa_{x_2,h,h'}$ on $\overline{U}_{x_1}\cap\overline{U}_{x_2}$, because

$$\sigma_i(x')\cdot\kappa_{x_1,h,h'}(x') = S(h)\circ S(h')\circ S(h\circ h')^{-1}(\sigma_i(x')) = \sigma_i(x')\cdot\kappa_{x_2,h,h'}(x')$$

for $x'\in\overline{U}_{x_1}\cap\overline{U}_{x_2}$. Now finitely many $U_{x_1},\ldots,U_{x_m}$ cover $\overline{V}_i$ and we thus see that

$$\omega(h,h')\circ\sigma_i = \operatorname{glue}(\kappa_{x_1,h,h'}|_{\overline{U}_{x_1}},\ldots,\kappa_{x_m,h,h'}|_{\overline{U}_{x_m}})$$

depends smoothly on h and h'.

To show the smoothness of ω_g, we derive an explicit formula for $\omega_g(g')\circ\sigma_i\in C^\infty(\overline{V}_i,K)$. Let $O_g\subseteq O$ be an open identity neighbourhood such that $g\circ O_g\circ g^{-1}\subseteq O$ and denote $\overline{g'}=g\circ g'\circ g^{-1}$ for $g'\in O_g$. Fix g' and $x\in\overline{V}_i$. Proceeding as in Remark 3.4.7 we find $j_\ell,\ldots,j_1$ such that

$$S(\overline{g'})^{-1}(\sigma_i(x)) = \sigma_{i_\ell}(\overline{g'}^{-1}(x))\cdot k_{j_\ell j_{\ell-1}}((\overline{g}_{j_{\ell-1}})^{-1}\circ\ldots\circ\left(\overline{g}_{j_1}\right)^{-1}(x))\cdot\ldots\cdot k_{j_1 i}(x).$$

Furthermore, let j'_1 be minimal such that

$$\big(F_M^{-1}\circ S(\overline{g'})_M^{-1}\big)(x) = g^{-1}\circ g'^{-1}(x)\in V_{j'_1}$$

and let U_x be an open neighbourhood of x (w.l.o.g. such that $\overline{U}_x \subseteq \overline{V}_i$ is a manifold with corners) such that $\overline{g'}^{-1}(\overline{U}_x) \subseteq V_{j_\ell}$ and $g^{-1} \circ g'^{-1}(\overline{U}_x) \subseteq V_{j'_1}$. Since $F_M = g$ and

$$F^{-1}(\sigma_{j_\ell}(\overline{g'}^{-1}(x'))) \in \sigma_{j'_1}(g^{-1} \circ g'^{-1}(x')) \text{ for } x' \in U_x$$

we have

$$F^{-1}\big(\sigma_{j_\ell}(\overline{g'}^{-1}(x'))\big) = \sigma_{j'_1}(g^{-1} \circ g'^{-1}(x')) \cdot k_{F,x,g'}(x') \text{ for } x' \in U_x,$$

for some smooth function $k_{F,x,g'} : U_x \to K$. In fact, we have

$$k_{F,x,g'}(x) = k_{\sigma_{j'_1}}(F^{-1}(\sigma_{j_\ell}(\overline{g'}^{-1}(x)))).$$

After possibly shrinking U_x, a construction as in Remark 3.4.2 shows that $k_{\sigma_{j'_1}} \circ F^{-1} \circ \sigma_{j_\ell}\big|_{\overline{U}_x}$ extends to a smooth function on M. Thus $k_{F,x,g'}|_{\overline{U}_x} \in C^\infty(\overline{U}_x, K)$ depends smoothly on g' for fixed x.

Accordingly, we find $j'_2, \ldots, j'_{\ell'}$ and a smooth function $k'_{F,x,g'} : \overline{U}_x \to K$ (possibly after shrinking U_x), depending smoothly on g such that

$$\begin{aligned} \omega_g(g')(\sigma_i(x)) = \sigma_i(x) \cdot \big[k'_{F,x,g'}(x) \cdot k_{j'_{\ell'} j'_{\ell'-1}}(g(x)) \cdot \ldots \cdot k_{j'_2 j'_1}(g'^{-1} \circ g^{-1}(x)) \cdot k_{F,x,g'}(x) \\ \cdot k_{j_\ell j_{\ell-1}}(g'(x)) \cdot \ldots \cdot k_{j_1 i}(x)\big]. \end{aligned} \tag{3.21}$$

Denote the element in brackets on the right hand side by $\kappa_{x,g'}$. Since we will not vary F and g in the sequel, we suppressed the dependence of $\kappa_{x,g'}$ on them. By continuity (cf. Remark 3.4.7), we find open neighbourhoods $O_{g'}$ and U'_x of g' and x (w.l.o.g. such that $\overline{U'}_x \subseteq \overline{V}_i$ is a manifold with corners) such that (3.21) defines $\omega_g(h')(\sigma_i(x'))$ for all $h' \in O_{g'}$ and $x' \in \overline{U}_x$. Then $\kappa_{x_1,g'} = \kappa_{x_2,g'}$ on $\overline{U}_{x_1} \cap \overline{U}_{x_2}$, finitely many $U_{x_1}, \ldots, U_{x_m}$ cover $\overline{V}_i$ and since the gluing and restriction maps from Lemma 2.2.20 and Proposition 2.2.21 are smooth,

$$\omega_g(g') \circ \sigma_i = \mathrm{glue}(\kappa_{x_1,g'}|_{\overline{U}_{x_1}}, \ldots, \kappa_{x_m,g'}|_{\overline{U}_{x_m}})$$

shows that $\omega_g(g') \circ \sigma_i$ depends smoothly on g'. ■

We thus have established the smoothness of the mappings T, ω and ω_g. As mentioned before, this will determine the smooth structure on $\mathrm{Aut}(\mathcal{P})$. We first give an description of the image of $\mathrm{Diff}(M)_{\mathcal{P}} := Q(\mathrm{Aut}(\mathcal{P}))$ in terms of $\mathcal{P}$, without referring to $\mathrm{Aut}(\mathcal{P})$.

Remark 3.4.13. Let $Q : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Diff}(M)$, $F \mapsto F_M$ be the homomorphism from Definition 3.1.1. If $g \in \mathrm{Diff}(M)_{\mathcal{P}}$, then there exists an $F \in \mathrm{Aut}(\mathcal{P})$ that covers g. Hence the commutative diagram

$$\begin{array}{ccccc} g^*(P) & \xrightarrow{g_{\mathcal{P}}} & P & \xrightarrow{F^{-1}} & P \\ {\scriptstyle g^*(\pi)}\big\downarrow & & {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ M & \xrightarrow{g} & M & \xrightarrow{g^{-1}} & M \end{array}$$

shows that $g^*(\mathcal{P})$ is equivalent to $\mathcal{P}$. On the other hand, if $\mathcal{P} \sim g^*(\mathcal{P})$, then the commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{\sim} & g^*(P) & \xrightarrow{g_{\mathcal{P}}} & P \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle g^*(\pi)}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ M & = & M & \xrightarrow{g} & M \end{array}$$

shows that there is an $F \in \mathrm{Aut}(\mathcal{P})$ covering g. Thus $\mathrm{Diff}(M)_{\mathcal{P}}$ consists of those diffeomorphisms preserving the equivalence class of $\mathcal{P}$ under pull-backs. This shows also that $\mathrm{Diff}(M)_{\mathcal{P}}$ is open because homotopic maps yield equivalent bundles. It thus is contained in $\mathrm{Diff}(M)_0$.

Note, that it is not possible to say what $\mathrm{Diff}(M)_{\mathcal{P}}$ is in general, even in the case of bundles over $M = \mathbb{S}^1$. In fact, we then have $\pi_0(\mathrm{Diff}(\mathbb{S}^1)) \cong \mathbb{Z}_2$ (cf. [Mi84]), and the component of $\mathrm{Diff}(\mathbb{S}^1)$, which does not contain the identity, are precisely the orientation reversing diffeomorphisms on $\mathbb{S}^1$. It follows from the description of the representing elements for bundles over $\mathbb{S}^1$ in Remark B.2.9 that pulling back the bundle along a orientation reversing diffeomorphism inverts the representing element in K. Thus we have $g^*(\mathcal{P}_k) \cong \mathcal{P}_{k^{-1}}$ for $g \notin \mathrm{Diff}(\mathbb{S}^1)_0$. If $\pi_0(K) \cong \mathbb{Z}_2$, then $\mathcal{P}_{k^{-1}}$ and $\mathcal{P}_k$ are equivalent because $[k] = [k^{-1}]$ in $\pi_0(K)$ and thus $g \in \mathrm{Diff}(\mathbb{S}^1)_{\mathcal{P}_k}$ and $\mathrm{Diff}(\mathbb{S}^1)_{\mathcal{P}_k} = \mathrm{Diff}(\mathbb{S}^1)$. If $\pi_0(K) \cong \mathbb{Z}_3$, then $\mathcal{P}_k$ and $\mathcal{P}_{k^{-1}}$ are *not* equivalent because $[k] \neq [k^{-1}]$ in $\pi_0(K)$ and thus $g \notin \mathrm{Diff}(\mathbb{S}^1)_{\mathcal{P}_k}$ and $\mathrm{Diff}(\mathbb{S}^1)_{\mathcal{P}_k} = \mathrm{Diff}(\mathbb{S}^1)_0$.■

Theorem 3.4.14 (Aut($\mathcal{P}$) as an extension of Diff(M)$_{\mathcal{P}}$ by Gau($\mathcal{P}$)). *Let $\mathcal{P}$ be a smooth principal K-bundle over the closed compact manifold M. If $\mathcal{P}$ has the property SUB, then* $\mathrm{Aut}(\mathcal{P})$ *carries a Lie group structure such that we have an extension of smooth Lie groups*

$$\mathrm{Gau}(\mathcal{P}) \hookrightarrow \mathrm{Aut}(\mathcal{P}) \xrightarrow{Q} \mathrm{Diff}(M)_{\mathcal{P}},$$

where $Q : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Diff}(M)$ *is the homomorphism from Definition 3.1.1 and* $\mathrm{Diff}(M)_{\mathcal{P}}$ *is the open subgroup of* $\mathrm{Diff}(M)$ *preserving the equivalence class of* $\mathcal{P}$ *under pull-backs.*

Proof. We identify $\mathrm{Gau}(\mathcal{P})$ with $C^\infty(P,K)^K$ and extend S to a (possibly non-continuous) section $S : \mathrm{Diff}(M)_{\mathcal{P}} \to \mathrm{Aut}(\mathcal{P})$ of Q. Now the preceding lemmas show that (T,ω) is a smooth factor system [Ne06a, Proposition II.8], which yields the assertion. However, we give an explicit description of the smooth structure by applying Proposition A.1.6, for which we have to check the assumptions. We introduce a smooth manifold structure on $W = C^\infty(P,K)^K \cdot S(O)$ by defining

$$\varphi : W \to C^\infty(P,K)^K \times O, \quad F \mapsto \big(F \cdot S(F_M)^{-1}, F_M\big)$$

to be a diffeomorphism. Let $O' \subseteq O$ be a symmetric open identity neighbourhood such that $O' \cdot O' \subseteq O$ and for each $g \in \mathrm{Diff}(M)$ denote by O_g the open identity neighbourhood from (3.20). Then multiplication in terms of φ is given by

$$(C^\infty(P,K)^K \times O')^2 \ni \big((\gamma,g),(\gamma' g')\big) \mapsto \varphi\big(\varphi^{-1}(\gamma,g)\cdot\varphi^{-1}(\gamma',g')\big) \in C^\infty(P,K)^K \times O,$$

inversion in terms of φ is given by

$$C^\infty(P,K)^K \times O \ni (\gamma, g) \mapsto \varphi(\varphi^{-1}(\gamma,g)^{-1}) \in C^\infty(P,K)^K \times O$$

and conjugation with $F \in \mathrm{Aut}(\mathcal{P})$ is given by

$$C^\infty(P,K)^K \times O_{Q(F)} \ni (\gamma, g) \mapsto \varphi\big(F \cdot \varphi^{-1}(\gamma,g) \cdot F^{-1}\big) \in C^\infty(P,K)^K \times O.$$

Now the smoothness of these maps follows with $\varphi^{-1}(\gamma,g) = F_\gamma \circ S(g)$ and $Q(S(g)) = g$ from Lemma 3.4.9, Lemma 3.4.12 and

$$\begin{aligned}
&\varphi\big(\varphi^{-1}(\gamma,g)\cdot\varphi^{-1}(\gamma',g')\big)\\
=&\varphi(F_\gamma \circ S(g) \circ F_{\gamma'} \circ S(g'))\\
=&\big(F_\gamma \circ S(g) \circ F_{\gamma'} \circ S(g') \circ S(g\circ g')^{-1}, g \circ g'\big)\\
=&\big(F_\gamma \circ \underbrace{S(g) \circ F_{\gamma'} \circ S(g)^{-1}}_{=T(\gamma,g)} \circ \underbrace{S(g) \circ S(g') \circ S(g\circ g')^{-1}}_{=\omega(g,g')}, g \circ g'\big)\\
&\varphi(\varphi^{-1}(\gamma,g)^{-1})\\
=&\big(S(g)^{-1} \circ F_{\gamma^{-1}} \circ S(g^{-1})^{-1}, g^{-1}\big)\\
=&\big(\underbrace{S(g)^{-1} \circ S(g^{-1})^{-1}}_{=\omega(g^{-1},g)^{-1}} \circ \underbrace{S(g^{-1}) \circ F_{\gamma^{-1}} \circ S(g^{-1})^{-1}}_{=T(\gamma^{-1},g^{-1})}, g^{-1}\big)\\
&\varphi\big(F \circ \varphi^{-1}(\gamma,g) \circ F^{-1}\big)\\
=&\big(F \circ \varphi^{-1}(\gamma,g) \circ F^{-1} \circ S(F_M \circ g \circ F_M^{-1})^{-1}, F_M \circ g \circ F_M^{-1}\big)\\
=&\big(\underbrace{F \circ F_\gamma \circ F^{-1}}_{c_F(\gamma)} \circ \underbrace{F \circ S(g) \circ F^{-1} \circ S(F_M \circ g \circ F_M^{-1})^{-1}}_{\omega_{F_M}(g)}, F_M \circ g \circ F_M^{-1}\big)
\end{aligned}$$

Since we have a smooth section $S : O \to \mathrm{Aut}(\mathcal{P})$, the quotient map

$$q : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Aut}(\mathcal{P})/C^\infty(P,K)^K \cong \mathrm{Diff}(M)_{\mathcal{P}}$$

defines the bundle projection of a smooth principal $C^\infty(P,K)^K$-bundle. ∎

Proposition 3.4.15. *In the setting of the previous theorem, the natural action*

$$\mathrm{Aut}(\mathcal{P}) \times P \to P, \quad (F,p) \mapsto F(p)$$

is smooth.

Proof. First we note the $\mathrm{Gau}(\mathcal{P}) \cong C^\infty(P,K)^K$ acts smoothly on P by $(\gamma,p) \mapsto p \cdot \gamma(p)$. Let $O \subseteq \mathrm{Diff}(M)$ be the open neighbourhood from Remark 3.4.4 and $S : O \to \mathrm{Aut}(P)$, $g \mapsto \widetilde{g_n} \circ \ldots \circ \widetilde{g_1}$ be the map from Definition 3.4.6. Then $\mathrm{Gau}(\mathcal{P}) \circ S(O)$ is an open neighbourhood in $\mathrm{Aut}(\mathcal{P})$ and it suffices to show that the restriction of the action to this neighbourhood is smooth due to Lemma A.3.3.

Since $\mathrm{Gau}(\mathcal{P})$ acts smoothly on $\mathcal{P}$, this in turn follows from the smoothness of the map

$$R : O \times P \to P, \quad (g,p) \mapsto S(g)(p) = \widetilde{g_n} \circ \ldots \circ \widetilde{g_1}(p).$$

In order to check the smoothness of R it suffices to check that $r_i : O \times P \to P \times O$, $(g,p) \mapsto (\widetilde{g_i}(p), g)$ is smooth, because then $R = \mathrm{pr}_1 \circ r_n \circ \ldots \circ r_1$ is smooth. Now the explicit formula

$$\widetilde{g_i}(\pi(p)) = \begin{cases} \sigma_i(g_i(\pi(p))) \cdot k_i(p) & \text{if } p \in \pi^{-1}(U_i) \\ p & \text{if } p \in \pi^{-1}(\overline{V}_i)^c \end{cases}$$

shows that r_i is smooth on $\big(O \times \pi^{-1}(U_i)\big) \cup \big(O \times \pi^{-1}(\overline{V_i})^c\big) = O \times P$. ∎

Remark 3.4.16. Of course, the Lie group structure on $\mathrm{Aut}(\mathcal{P})$ from Theorem 3.4.14 depends on the choice of S and thus on the choice of the chart $\varphi : O \to \mathcal{V}(M)$ from Remark 3.4.4, the choice of the trivialising system from Remark 3.4.2 and the choice of the partition of unity chosen in the proof of Lemma 3.4.5.

However, different choices lead to isomorphic Lie group structures on $\mathrm{Aut}(\mathcal{P})$ and, moreover to equivalent extensions. To show this we show that $\mathrm{id}_{\mathrm{Aut}(\mathcal{P})}$ is smooth when choosing two different trivialising systems $\overline{\mathcal{V}} = (\overline{V}_i, \sigma_i)_{i=1,\ldots,n}$ and $\overline{\mathcal{V}}' = (\overline{V}'_j, \tau_j)_{j=1,\ldots,m}$.

Denote by $S : O \to \mathrm{Aut}(\mathcal{P})$ and $S' : O \to \mathrm{Aut}(\mathcal{P})$ the corresponding sections of Q. Since

$$\mathrm{Gau}(\mathcal{P}) \circ S(O) = Q^{-1}(O) = \mathrm{Gau}(\mathcal{P}) \circ S'(O)$$

is an open unit neighbourhood and $\mathrm{id}_{\mathrm{Aut}(\mathcal{P})}$ is an isomorphism of abstract groups, it suffices to show that the restriction of $\mathrm{id}_{\mathrm{Aut}(\mathcal{P})}$ to $Q^{-1}(O)$ is smooth. Now the smooth structure on $Q^{-1}(O)$ induced from S and S' is given by requiring

$$Q^{-1}(O) \ni F \mapsto (F \circ S(F_M)^{-1}, F_M) \in \mathrm{Gau}(\mathcal{P}) \times \mathrm{Diff}(M)$$
$$Q^{-1}(O) \ni F \mapsto (F \circ S'(F_M)^{-1}, F_M) \in \mathrm{Gau}(\mathcal{P}) \times \mathrm{Diff}(M)$$

to be diffeomorphisms and we thus have to show that

$$O \ni g \mapsto S(g) \circ S'(g)^{-1} \in \mathrm{Gau}(\mathcal{P})$$

is smooth. By deriving explicit formulae for $S(g) \circ S'(g)^{-1}(\sigma_i(x))$ on a neighbourhood U_x of $x \in \overline{V}_i$, and O_g of $g \in O$ this follows exactly as in Lemma 3.4.12. ∎

Remark 3.4.17. A Lie group structure on $\mathrm{Aut}(\mathcal{P})$ has been considered in [ACMM89] in the convenient setting, and the interest in $\mathrm{Aut}(\mathcal{P})$ as a symmetry group coupling the gauge symmetry of Yang-Mills theories and the $\mathrm{Diff}(M)$-invariance of general relativity is emphasised. Moreover, it is also shown that $\mathrm{Gau}(\mathcal{P})$ is a split Lie subgroup of $\mathrm{Aut}(\mathcal{P})$, that

$$\mathrm{Gau}(\mathcal{P}) \hookrightarrow \mathrm{Aut}(\mathcal{P}) \twoheadrightarrow \mathrm{Diff}(M)_{\mathcal{P}}$$

is an exact sequence of Lie groups and that the action $\mathrm{Aut}(\mathcal{P}) \times P \to P$ is smooth. However, the Lie group structure is constructed out of quite general arguments allowing to give the space $\mathrm{Hom}(\mathcal{P}, \mathcal{P})$ of bundle morphisms a smooth structure and then to consider $\mathrm{Aut}(\mathcal{P})$ as an open subset of $\mathrm{Hom}(\mathcal{P}, \mathcal{P})$.

The approach taken in this section is somehow different, since the Lie group structure on $\mathrm{Aut}(\mathcal{P})$ is constructed by foot and the construction provides explicit charts given by charts of $\mathrm{Gau}(\mathcal{P})$ and $\mathrm{Diff}(M)$. ■

Remark 3.4.18. The approach to the Lie group structure in this section used detailed knowledge on the chart $\varphi : O \to \mathcal{V}(M)$ of the Lie group $\mathrm{Diff}(M)$ from Remark 3.4.4. We used this when decomposing a diffeomorphism into a product of diffeomorphisms with support in some trivialising subset of M. The fact that we needed was that for a diffeomorphism $g \in O$ we have $g(m) = m$ if the vector field $\varphi(g)$ vanishes in m. This should also be true for the charts on $\mathrm{Diff}(M)$ for compact manifolds with corners and thus the procedure of this section should carry over to bundles over manifolds with corners. ■

Remark 3.4.19. In some special cases, the extension $\mathrm{Gau}(\mathcal{P}) \hookrightarrow \mathrm{Aut}(\mathcal{P}) \twoheadrightarrow \mathrm{Diff}(M)_{\mathcal{P}}$ from Theorem 3.4.14 splits. This is the case for trivial bundles and for bundles with abelian structure group K, but also for frame bundles, since we then have a natural homomorphism $\mathrm{Diff}(M) \to \mathrm{Gau}(\mathcal{P})$, $g \mapsto dg$. However, it would be desirable to have a characterisation of the bundles, for which this extension splits. ■

Problem 3.4.20. Find a characterisation of those principal K-bundles $\mathcal{P}$ for which the extension $\mathrm{Gau}(\mathcal{P}) \hookrightarrow \mathrm{Aut}(\mathcal{P}) \twoheadrightarrow \mathrm{Diff}(M)_{\mathcal{P}}$ splits on the group level. ■

Chapter 4

Calculating homotopy groups of gauge groups

As indicated in Appendix A and Section 5.2, a good understanding of the low-dimensional homotopy groups of an infinite-dimensional Lie group is an important key to their Lie theory. In particular, the first and second (rational) homotopy groups are important when considering central extensions of connected Lie groups.

In this chapter we illustrate how one can access the (rational) homotopy groups of gauge groups. Due to the weak homotopy equivalence

$$\pi_n(\mathrm{Gau}(\mathcal{P})) \cong \pi_n(\mathrm{Gau}_c(\mathcal{P}))$$

from Theorem 3.2.13 we may restrict our attention to continuous gauge groups. This makes life easier since continuous functions are much more flexible than smooth functions. The main tool will be the evaluation fibration and the resulting long exact homotopy sequence introduced in the first section.

Of particular interest will be principal bundles over spheres and compact, closed surfaces, because they are the the easiest non-trivial examples but already cover many interesting cases. In particular, the case of bundles over $\mathbb{S}^1$ will become important in Chapter 5. Note that bundles over orientable, but non-compact or non-closed surfaces with connected structure group are always trivial (cf. Proposition B.2.10).

Throughout this chapter we will consider continuous principal bundles and identify the continuous gauge group $\mathrm{Gau}_c(\mathcal{P})$ with the space of K-equivariant continuous mappings $C(P,K)^K$. To avoid confusion with the homotopy groups, we furthermore denote the bundle projection of the principal bundle $\mathcal{P} = (K, \eta : P \to M)$ with η instead of π.

4.1 The evaluation fibration

Let $\mathcal{P}$ be a continuous principal bundle. In this section we study the evaluation fibration

$$\mathrm{ev} : C(P,K)^K \to K, \quad \gamma \mapsto \gamma(p_0),$$

where p_0 is the base-point of P. Under some mild restrictions it will turn out to be a Serre fibration and thus leads to a long exact sequence for the homotopy groups of $C(P,K)^K$.

Definition 4.1.1 (Evaluation fibration). If $\mathcal{P}$ is a continuous principal K-bundle and $p_0 \in P$ denotes the base-point, then the map $\mathrm{ev} : C(P,K)^K \to K$, $\gamma \mapsto \gamma(p_0)$ is called the *evaluation fibration*. The kernel

$$C_*(P,K)^K := \ker(\mathrm{ev}) = \{\gamma \in C^\infty(P,K)^K : \gamma(p_0) = e\}$$

is called the *pointed gauge group*. Note that each $\gamma \in C_*(P,K)^K$ vanishes on the whole fibre $p_0 \cdot K$ through p_0, because we have $\gamma(p_0 \cdot k) = k^{-1} \cdot \gamma(p_0) \cdot k^{-1} = e$. ■

Lemma 4.1.2. *If K is a locally contractible topological group and $\mathcal{P} = (K, \eta : P \to M)$ is a continuous principal K-bundle over the finite-dimensional manifold with corners M, then the evaluation fibration from Definition 4.1.1 defines an extension of topological groups*

$$C_*(P,K)^K \xhookrightarrow{\iota} C(P,K)^K \xrightarrow{\mathrm{ev}} K,$$

which has continuous local sections. In particular, it is a Serre fibration in the sense of [Br93, Chapter VII.6] and induces a long exact homotopy sequence

$$\begin{aligned} \ldots\ldots \to \pi_{n+1}(K) \xrightarrow{\delta_{n+1}} \pi_n(C_*(P,K)^K) \xrightarrow{\pi_n(\iota)} \pi_n(C(P,K)^K) \\ \xrightarrow{\pi_n(\mathrm{ev})} \pi_n(K) \xrightarrow{\delta_n} \pi_{n-1}(C_*(P,K)^K) \to \ldots . \end{aligned} \tag{4.1}$$

Proof. If suffices to construct a continuous local section $\sigma : W \to C(P,K)^K$ of ev for some open unit neighbourhood $W \subseteq K$, since then $\mathrm{ev} : C(P,K)^K \to K$ is a locally trivial bundle and thus a locally trivial fibration (cf. [Br93, Corollary VII.6.12]). Since K is locally contractible, there exist open unit neighbourhoods W, W' and a continuous map $F : [0,1] \times W \to W'$ such that $F(0,k) = e$, $F(1,k) = k$ for all $k \in W$ and $F(t,e) = e$ for all $t \in [0,1]$. For $k \in W$, we set $\tau_k := F(\cdot, k)$, which is a continuous path and satisfies $\tau_k(0) = e$ and $\tau_k(1) = k$.

Now let m_0 be the base-point in M and let $U \subseteq M$ be an open neighbourhood of m_0 such that there exists a chart

$$\varphi : U \to \varphi(U) \subseteq \mathbb{R}^{n,r}_+ := \mathbb{R}^{n-r} \times \mathbb{R}^r_+$$

with $\varphi(m_0) = 0$ and a continuous section $\sigma : U \to P$ with $k_\sigma : \eta^{-1}(U) \to K$, determined by $\sigma(\eta(p)) \cdot k_\sigma(p) = p$. Then there exists an $\varepsilon > 0$ such that

$$\mathbb{R}_+^{n,r} \cap B_\varepsilon(0) := \{x \in \mathbb{R}_+^{n,r} : \|x\| \leq \varepsilon\} \subseteq \varphi(U)$$

and

$$\gamma_k(p) = \begin{cases} k_\sigma(p)^{-1} \cdot \tau_k(1 - \varepsilon^{-1}\|\varphi(\eta(p))\|) \cdot k_\sigma(p) & \text{if } p \in (\varphi \circ \eta)^{-1}(B_\varepsilon(0)) \\ e & \text{if } p \notin (\varphi \circ \eta)^{-1}(B_\varepsilon(0)) \end{cases}$$

defines an element of $C(P,K)^K$, because $\tau_k(0) = e$ for all $k \in W$ and thus $\gamma_k(p) = e$ if $p \in \partial((\varphi \circ \eta)^{-1}(B_\varepsilon(0)))$. Furthermore, τ_k depends continuously on k by the exponential law and so does γ_k. Eventually,

$$W \ni k \mapsto \gamma_k \in C(P,K)^K$$

defines a continuous section of ev. ■

The idea of this chapter is to consider bundles for which the homotopy groups of the pointed gauge group $\pi_n(C_*(P,K)^K)$ are well accessible. Then the previous Lemma leads to a long exact homotopy sequence that one can use to get information on $\pi_n(C(P,K)^K)$. In particular, this will turn out to be the case for bundles over spheres and compact, closed and orientable surfaces. In these cases, $\pi_n(C_*(P,K)^K)$ can be expressed in terms of the homotopy groups $\pi_n(K)$ and, moreover, one can also calculate the connecting homomorphisms. To motivate this idea we first consider the case of trivial bundles over spheres and recall some facts on $\pi_n(K)$ for finite-dimensional K.

Lemma 4.1.3. *If $P = \mathbb{S}^m \times K$ is the trivial bundle over $\mathbb{S}^m$ and $n \geq 1$, then*

$$\pi_n(C(P,K)^K) \cong \pi_{n+m}(K) \oplus \pi_n(K).$$

Proof. For trivial bundles we have a globally defined continuous section and thus Remark 3.2.1 yields $C(P,K)^K \cong C(\mathbb{S}^m, K)$. Now $C(\mathbb{S}^m,K) \cong C_*(\mathbb{S}^m,K) \rtimes K$ and thus

$$\pi_n(C(P,K)^K) \cong \pi_n(C(\mathbb{S}^m,K)) \cong \pi_n(C_*(\mathbb{S}^m,K)) \oplus \pi_n(K).$$

Now the assertion follows from

$$\begin{aligned}\pi_n(C_*(\mathbb{S}^m,K)) &\cong \pi_0(C_*(\mathbb{S}^n, C_*(\mathbb{S}^m,K))) \\ &\cong \pi_0(C_*(\mathbb{S}^n \wedge \mathbb{S}^m, K)) \cong \pi_0(C_*(\mathbb{S}^{n+m},K)) \cong \pi_{n+m}(K).\end{aligned}$$ ■

Remark 4.1.4. We recall some facts on the homotopy groups of a finite-dimensional Lie group K. One important fact is that $\pi_2(K)$ always vanishes [Mi95, Theorem 3.7]. Furthermore, we have $\pi_3(K) \cong \mathbb{Z}$ if K has a compact Lie algebra [Mi95, Theorem 3.8], because then K_0 is compact [DK00, Corollary 3.6.3] and we have $\pi_3(K) = \pi_3(K_0)$. Furthermore, in [Mi95] one finds a table with $\pi_n(K)$ up to $n = 15$, showing in particular $\pi_4(\mathrm{SU}_2(\mathbb{C})) \cong \pi_5(\mathrm{SU}_2(\mathbb{C})) \cong \mathbb{Z}_2$ and $\pi_6(\mathrm{SU}_2(\mathbb{C})) \cong \mathbb{Z}_{12}$. ■

We want to reduce the determination of $\pi_n(C_*(P,K)^K)$ to the determination of $\pi_n(C_*(M,K))$. We will first observe that we have $\pi_n(C_*(P,K)^K) \cong \pi_n(C_*(M,K))$ if one considers bundles with the property that the restriction to the complement of the base-point is trivial and to functions not only vanishing in base-points but also on a whole neighbourhood of them. This covers the cases of bundles that we are aiming for, and it will show up later that the mapping spaces are homotopically equivalent if the neighbourhood of the base-point is chosen appropriately.

Definition 4.1.5. If (X,x_0) and (Y,y_0) are pointed topological spaces and $A \subseteq X$, is a subset containing x_0, then we denote by

$$C_A(X,Y) := \{f \in C(X,Y) : f(A) = \{y_0\}\} \subseteq C_*(X,Y)$$

the space of continuous functions mapping A to the base point in Y. ■

Lemma 4.1.6. *If $\mathcal{P}$ is a continuous principal K-bundle over the regular space X, x_0 is the base point of X such that $X \backslash \{x_0\}$ is trivialising, then for each open neighbourhood $U \subseteq X$ of x_0 there is an isomorphism of topological groups*

$$C_{\eta^{-1}(\overline{U})}(P,K)^K \xrightarrow{\cong} C_{\overline{U}}(X,K), \quad f \mapsto \widetilde{f \circ \sigma},$$

where $\sigma : X \backslash \{x_0\} \to P$ is a continuous section and $\widetilde{f \circ \sigma}$ is the continuous extension of $f \circ \sigma$ to X by e in x_0.

Proof. Let $(U_1, \sigma_1, U_2 \sigma_2)$ be an continuous open trivialising system with $U_1 \subseteq \overline{U}$, $U_2 = X \backslash \{x_0\}$ and $k_{12} : U_1 \cap U_2 \to K$ be the corresponding transition function (cf. Remark B.1.7). Then Remark 3.2.1 yields

$$\begin{aligned} C(P,K)^K \cong G_{\mathcal{U}}(\mathcal{P}) = \{&(\gamma_1,\gamma_2) \in C(U_1,K) \times C(U_2,K) : \\ &\gamma_1(x) = k_{12}(x) \cdot \gamma_2(x) \cdot k_{21}(x) \text{ for all } x \in U_1 \cap U_2\}, \end{aligned}$$

where the isomorphism is given by $f \mapsto (f \circ \sigma_1, f \circ \sigma_2)$. This isomorphism in turn induces

$$C_{\eta^{-1}(\overline{U})}(P,K)^K \cong G_{\mathcal{U},\overline{U}}(\mathcal{P}) := \{(\gamma_1,\gamma_2) \in G_{\mathcal{U}}(\mathcal{P}) : \gamma_1 \equiv e \text{ and } \gamma_2|_{U_2 \cap \overline{U}} \equiv e\},$$

because $\sigma_1(U_1) \subseteq \eta^{-1}(\overline{U})$ implies $f(\sigma_1(x)) = e$ and $\sigma_2(x) \in \eta^{-1}(\overline{U}) \Leftrightarrow x \in \overline{U} \cap U_2$. Now

$$C_{\overline{U}}(X,K) \to G_{\mathcal{U},\overline{U}}(\mathcal{P}), \quad f \mapsto (f|_{U_1}, f|_{U_2}) \tag{4.2}$$

is an isomorphism of abstract groups which is continuous. To construct the inverse isomorphism we note that if $(\gamma_1,\gamma_2) \in G_{\mathcal{U}}(\mathcal{P})$ and $\gamma_1 \equiv e$, then we can extend γ_2 to $\widetilde{\gamma}_2 : M \to K$ by $\widetilde{\gamma}_2(x_0) = e$, because γ_2 vanishes on the neighbourhood U_1 of x_0. Since X is assumed to be regular, there exists a closed subset $C \subseteq U$ with $x_0 \in C$, and a direct verification in the compact-open topology shows that the map

$$G_{\mathcal{U},\overline{U}}(\mathcal{P}) \to C_{\overline{U}}(X,K), \quad (\gamma_1,\gamma_2) \mapsto \widetilde{\gamma}_2$$

is continuous. Since it is the inverse to (4.2), this establishes the assertion. ■

According to the previous Lemma, we now want to replace $C_*(P,K)^K$ by a homotopically equivalent space of gauge transformations vanishing on a suitable neighbourhood of $\eta^{-1}(x_0)$. To make this precise we shall need a concept to "localise" homotopy equivalences, obtained from collapsing subspaces, that become constant outside some neighbourhood of the subspace. This motivates the following definition.

Definition 4.1.7. Let X be a topological space, x_0 be its base-point, and U_0, U_1 be open neighbourhoods of x_0 with $\overline{U}_0 \subseteq U_1$. Then a continuous map $R : [0,1] \times X \to X$ is called a *strong retraction of* $\overline{U}_0$ *to* x_0 *relative to* $X\backslash U_1$ if $R(0,\cdot) = \mathrm{id}_X$, $R(t,U_0) \subseteq U_0$, $R(t,U_1) \subseteq U_1$, $R(1,\overline{U}_0) = \{x_0\}$ and $R(t,x) = x$ for all $t \in [0,1]$ and $x \notin U_1$. This is a homotopy from the identity $R(0,\cdot)$ to a map $R(1,\cdot)$, which collapses $\overline{U}_0$ to x_0 and is the identity on the larger set $X\backslash U_1$. ■

Note that the previous definition is slightly different from the requirements that $\overline{U}_0$ is contractible and $\overline{U}_0 \hookrightarrow X$ is a cofibration. These requirements would only yield the homotopy R *without* the requirement that $R(t,\cdot)$ is the identity on some larger set. This property will become important in the sequel, because it enables us to lift these homotopies to equivariant homotopies on the bundles.

Lemma 4.1.8. *If M is a finite-dimensional manifold with corners and m_0 is its base-point, then for each open neighbourhood $V \subseteq M$ of m_0, there exist neighbourhoods $U_0, U_1 \subseteq V$, such that there exists a strong retraction R of $\overline{U}_0$ to m_0 relative to $M\backslash U_1$.*

In particular, if $M = \mathbb{S}^m$ and U_S, x_S and x_N are as in Remark B.2.9, then we can choose U_0 and U_1 such that $U_0 = U_S$ and $\overline{U}_1 \subseteq \mathbb{S}^m\backslash\{x_N\}$.

Proof. Let $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n_+$ be a chart around m_0 and let U_0 and U_1 be open neighbourhoods of m_0 in $V \cap U$ such that $\overline{U}_0 \subseteq U_1$ and $\varphi(\overline{U}_0)$ and $\varphi(\overline{U}_1)$ are convex. Furthermore, let $\lambda : M \to [0,1]$ be smooth with $\operatorname{supp}(\lambda) \subseteq U_1$ and $\lambda \equiv 1$ on a neighbourhood of $\overline{U}_0$. Set

$$R : [0,1] \times M \to M,\ (t,x) \mapsto \begin{cases} \varphi^{-1}\big((1-t\cdot\lambda(x))\varphi(x) + t\cdot\lambda(x)\cdot\varphi(m_0)\big) & \text{if } x \in U \\ x & \text{if } x \notin U. \end{cases}$$

Then $\operatorname{supp}(\lambda) \subseteq U_1 \subseteq U_1$ implies that R is continuous and $R(t,x) = x$ if $x \notin U_1$. Furthermore, we have $R(0,\cdot) = \mathrm{id}_M$ and $\lambda|_{\overline{U_0}} \equiv 1$ implies $R(1,\overline{U}_0) = \{m_0\}$. Since $\overline{U}_0$ and $\overline{U}_1$ are convex, we also have $R(t,\overline{U}_0) \subseteq \overline{U}_0$ and $R(t,\overline{U_1}) \subseteq \overline{U}_1$. ■

As indicated before, the group of gauge transformations, vanishing on a suitable neighbourhood of the fibre through p_0, is homotopy equivalent to the pointed gauge group $C_*(P,K)^K$. We first consider the case of trivial bundles, where we have $C(P,K)^K \cong C(M,K)$.

Lemma 4.1.9. *If X, Y are topological spaces, X is locally compact and $R : [0,1] \times X \to X$ is a strong retraction of $\overline{U_0}$ to x_0 relative to $X \backslash U_1$, then the inclusion*

$$C_{\overline{U}_0}(X,Y) \xhookrightarrow{\iota} C_*(X,Y)$$

is a homotopy equivalence.

Proof. Since $R(0,\cdot) = \mathrm{id}_X$, we may write ι as the pull-back $R(0,\cdot)^*$. Since $R(1,\cdot)(\overline{U_0}) = \{x_0\}$, we get a continuous map $R(1,\cdot)^* : C_*(X,Y) \to C_{\overline{U}_0}(X,Y)$. Since $R(1,\cdot)$ is homotopic to $R(0,\cdot)$, we have

$$R(0,\cdot)^* \circ R(1,\cdot)^* \simeq R(0,\cdot)^* \circ R(0,\cdot)^* = \mathrm{id}_{C_*(X,Y)},$$
$$R(1,\cdot)^* \circ R(0,\cdot)^* \simeq R(0,\cdot)^* \circ R(0,\cdot)^* = \mathrm{id}_{C_{\overline{U}_0}(X,Y)},$$

and thus $R(1,\cdot)^*$ is a homotopy inverse to $R(0,\cdot)^*$. ■

Proposition 4.1.10. *Let $\mathcal{P} = (K, \eta : P \to M)$ be a continuous principal K-bundle over the finite-dimensional manifold with corners M, and let V be a trivialising open neighbourhood of the base-point m_0. If $R : [0,1] \times X \to X$ is a strong retraction of $\overline{U_0}$ to m_0 relative to $X \backslash U_1$ and $\overline{U}_1 \subseteq V$, then the inclusion*

$$C_{\eta^{-1}(\overline{U}_0)}(P,K)^K \hookrightarrow C_*(P,K)^K$$

is is a homotopy equivalence.

Proof. Let $\sigma : V \to P$ be a continuous section, defining $k_\sigma : \eta^{-1}(V) \to K$ by $p = \sigma(\eta(p)) \cdot k_\sigma(p)$. Then

$$R_{\mathcal{P}} : [0,1] \times P \to P, \quad (t,p) \mapsto \begin{cases} \sigma(R(t,\eta(p))) \cdot k_\sigma(p) & \text{if } \eta(p) \in V \\ p & \text{if } \eta(p) \notin \overline{U}_1 \end{cases}$$

is well-defined, because $R(t,m) = m$ if $m \notin U_1$. Thus the map $R_{\mathcal{P}}$ is continuous and $R_{\mathcal{P}}(t,\cdot)$ is K-equivariant, because for $\eta(p) \in V$ we have

$$R_{\mathcal{P}}(t, p \cdot k) = \sigma(R(t,\eta(p))) \cdot k_\sigma(p \cdot k) = \sigma(R(t,\eta(p))) \cdot k_\sigma(p) \cdot k = R_{\mathcal{P}}(t,p) \cdot k,$$

since $k_\sigma(p \cdot k) = k_\sigma(p) \cdot k$ if $\eta(p) \in V$. Furthermore, $R_{\mathcal{P}}(0,\cdot) = \mathrm{id}_P$ and thus the inclusion may be written as the push-forward $R_{\mathcal{P}}(0,\cdot)^*$. Now $R_{\mathcal{P}}(1, \eta^{-1}(\overline{U}_0)) \subseteq \eta^{-1}(x_0)$ and thus $f(R_{\mathcal{P}}(1,\cdot))$ vanishes on $\eta^{-1}(\overline{U}_0)$ if $f \in C_*(P,K)^K$. Since

$$R_{\mathcal{P}}(1,\cdot)^* \circ R_{\mathcal{P}}(0,\cdot)^* \simeq R_{\mathcal{P}}(0,\cdot)^* \circ R_{\mathcal{P}}(0,\cdot)^* = \mathrm{id}_{C_{\eta^{-1}(\overline{U}_0)}(P,K)^K}$$

and

$$R_{\mathcal{P}}(0,\cdot)^* \circ R_{\mathcal{P}}(1,\cdot)^* \simeq R_{\mathcal{P}}(0,\cdot)^* \circ R_{\mathcal{P}}(0,\cdot)^* = \mathrm{id}_{C_*(P,K)^K},$$

we have that $R_{\mathcal{P}}(1,\cdot)^*$ is a homotopy inverse to $R_{\mathcal{P}}(0,\cdot)^*$ and thus that the inclusion is a homotopy equivalence. ■

We collect the information we have so far for bundles over spheres in the following proposition. We will throughout this section use the notation for spheres introduced in Remark B.2.9.

Proposition 4.1.11. *Let $\mathcal{P} = (K, \eta : P \to \mathbb{S}^m)$ be a continuous principal K-bundle and K be locally contractible. Then there exists a strong retraction of $\overline{U}_S$ to x_S relative to to $\mathbb{S}^m \backslash U_1$ for some $U_1 \supseteq \overline{U_S}$ with $x_N \notin \overline{U_1}$. Furthermore, we have the homotopy equivalences*

$$C_*(P,K)^K \simeq C_{\eta^{-1}(\overline{U}_S)}(P,K)^K \cong C_{\overline{U}_S}(\mathbb{S}^m, K) \simeq C_*(\mathbb{S}^m, K)$$

from Proposition 4.1.10, Lemma 4.1.6 and Lemma 4.1.9 inducing

$$\pi_n(C_*(P,K)^K) \cong \pi_n(C_*(\mathbb{S}^m, K)) \cong \pi_{n+m}(K).$$

With respect to this isomorphism, the long exact homotopy sequence of the evaluation fibration (4.1) *becomes*

$$\cdots \to \pi_{n+1}(K) \xrightarrow{\delta_{n+1}} \pi_{n+m}(K) \to \pi_n(C(P,K)^K) \to \pi_n(K) \xrightarrow{\delta_n} \pi_{n+m-1}(K) \to \cdots . \tag{4.3}$$

■

In order to perform a similar construction for bundles over compact, closed and orientable surfaces we need more information on the algebraic topology of these surfaces and the corresponding mapping groups.

Remark 4.1.12. Recall the notation for closed, compact and orientable surfaces from Remark B.2.11. The identification $A_{2g} \cong \partial\mathbb{B}^2$ shows in particular that if X is an arbitrary topological space, then a map $f : A_{2g} \to X$ extends to a continuous map $f : \Sigma \to X$ if and only if it extends to $\operatorname{int}(\mathbb{B}^2)$ an thus is zero-homotopic. This can be expressed by the property that $\pi_1(f) : \pi_1(A_{2g}) \to \pi_1(X)$ annihilates the commutator (B.7) in Remark B.2.11 and hence factors through a homomorphism $\pi_1(\Sigma) \cong \mathbb{Z}^{2g} \to \pi_1(X)$.

Furthermore, if such a homomorphism $\pi_1(\Sigma) \to \pi_1(X)$ is given, then we lift it to a homomorphism $\pi_1(A_{2g}) \to \pi_1(X)$, which can be represented by a map $A_{2g} \to X$. Since this map extends to Σ, we have shown that

$$C_*(\Sigma, X) \to \operatorname{Hom}(\pi_1(\Sigma), \pi_1(X)), \quad f \mapsto \pi_1(f)$$

is surjective.

Now, consider for fixed $j \leq 2g$ the homomorphism $\mathbb{Z}^{2g} \to \mathbb{Z}$, given on the generators by δ_{ij}. If we take $X = \mathbb{S}^1$, then the preceding implies that we obtain continuous maps $\chi_j : \Sigma \to \mathbb{S}^1$ such that $\pi_1(\chi_j)([\alpha_i]) = \delta_{ij}$. We can even arrange χ_j such that

$$\chi_j \circ \alpha_i = \begin{cases} \mathrm{id}_{\mathbb{S}^1} & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases} \tag{4.4}$$

■

if we start with the continuous map $\chi_j^0 : A_{2g} \to \mathbb{S}^1$ which is on $\mathbb{S}^1_j$ the identification with $\mathbb{S}^1$ and constantly e otherwise. Clearly, $\pi_1(\chi_j)$ annihilates the commutator in (B.7) and thus extends to Σ.

Remark 4.1.13. We recall that if X is a space and $A \subseteq X$, then there is a bijection between the continuous functions on X/A and the continuous functions on X that are constant (cf. [Bo89a, §I.3.4]). For any other space Y this bijection is given by the continuous map $q^* : C_*(X/A, Y) \to C_A(X, Y)$, $f \mapsto f \circ q$, where $q : X \to X/A$ is the quotient map. Moreover, if A is closed, then a direct verification in the compact-open topology shows that this map is also open and thus $C(X/A, Y)$ and $C_A(X, Y)$ are also homeomorphic.

In particular, if Σ is a compact, closed and orientable surface and K is a topological group, then

$$C_{A_{2g}}(\Sigma, K) \cong C_*(\mathbb{S}^2, K),$$

and furthermore we have

$$\pi_n(C_{A_{2g}}(\Sigma, K)) \cong \pi_n(C_*(\mathbb{S}^2, K)) \cong \pi_{n+2}(K).$$ ■

We now show that these information lead to a precise description of the pointed mapping group $C_*(\Sigma, K)$ in terms of $C_*(\mathbb{S}^2, K)$ and $C_*(\mathbb{S}^1, K)$. Note that this is exactly what we are aiming for, because $C_*(\Sigma, K)$ is homotopy equivalent to $C_*(P, K)^K$, and we thus obtain a precise description of $C_*(P, K)^K$ in terms of the homotopy groups of K.

Lemma 4.1.14. *Let Σ be a compact closed and orientable surface and K be a topological group and consider*

$$r : C_*(\Sigma, K) \to C_*(\mathbb{S}^1, K)^{2g}, \quad f \mapsto (f \circ \alpha_1, \ldots, f \circ \alpha_{2g}).$$

This map has $C_{A_{2g}}(\Sigma, K) \cong C_*(\mathbb{S}^2, K)$ as kernel, and with respect to this identification the exact sequence

$$C_*(\mathbb{S}^2, K) \hookrightarrow C_*(\Sigma, K) \xrightarrow{r} C_*(\mathbb{S}^1, K)^{2g} \tag{4.5}$$

has a globally defined continuous (but non-homomorphic) section. In particular, $C_*(\Sigma, K)$ is homeomorphic to $C_*(\mathbb{S}^2, K) \times C_*(\mathbb{S}^1, K)^{2g}$.

Proof. The kernel of r is in fact $C_{A_{2g}}(\Sigma, K)$, because $f \circ \alpha_i$ vanishes if and only if f vanishes on $\mathbb{S}^1_i$ and $A_{2g} = \bigcup_i \mathbb{S}^1_i$. Furthermore, $C_{A_{2g}}(\Sigma, K) \cong C_*(\mathbb{S}^2, K)$ by Remark 4.1.13.

A continuous inverse to r can be constructed as follows. Let $\chi_j : \Sigma \to \mathbb{S}^1$ be the continuous maps constructed in Remark 4.1.12. Then we define

$$C_*(\mathbb{S}^1, K)^{2g} \to C_*(\Sigma, K), \quad (f_1, \ldots, f_{2g}) \mapsto \prod_{i=j}^{2g} f_j \circ \chi_j$$

This is in fact a section of r, because (4.4) implies

$$\prod_{j=1}^{2g}(f\circ\chi_j\circ\alpha_i)(m)=f\circ\chi_i\circ\alpha_i(m)=f(m)\ \text{ for }\ m\in\mathbb{S}^1.$$

Now the existence of a continuous section implies that $C_*(\Sigma,K)$ is a trivial principal $C_*(\mathbb{S}^2,K)$-bundle over $C_*(\mathbb{S}^1,K)^{2g}$, and thus $C_*(\Sigma,K)$ is isomorphic as a $C_*(\mathbb{S}^2,K)$-space to $C_*(\mathbb{S}^2,K)\times C_*(\mathbb{S}^1,K)^{2g}$. ■

For bundles over compact, closed and orientable surfaces with connected structure group, the above considerations now lead to a similar long exact sequence for $\pi_n(C(P,K)^K)$ as in the case of bundles over spheres in Proposition 4.1.11.

Proposition 4.1.15. *Let $\mathcal{P}=(K,\eta:P\to M)$ be a continuous principal K-bundle over a compact, closed and orientable surface Σ and let K be connected and locally contractible. Furthermore, set $U_\Sigma:=\Sigma\backslash B_{\frac{1}{2}}(0)$ (where we identify Σ with a quotient of $\mathbb{B}^2$ as in Remark B.2.11).*

Then there exists a strong retraction of $\overline{U}_0$ to the base-point x_0 of $A_{2g}\subseteq\Sigma$ relative to to $\Sigma\backslash U_1$ for some $U_0,U_1\subseteq\Sigma$ with $\overline{U}_1\subseteq U_\Sigma$. Furthermore, we have the homotopy equivalences

$$C_*(P,K)^K\simeq C_{\eta^{-1}(\overline{U}_0)}(P,K)^K\cong C_{\overline{U}_0}(\Sigma,K)\simeq C_*(\Sigma,K)\simeq C_*(\mathbb{S}^2,K)\times C_*(\mathbb{S}^1,K)^{2g}$$

from Proposition 4.1.10, Lemma 4.1.6, Lemma 4.1.9 and Lemma 4.1.14 inducing for $n\geq 1$

$$\pi_n(C_*(P,K)^K)\cong\pi_n(C_*(\Sigma,K))\cong\pi_{n+2}(K)\oplus\pi_{n+1}(K)^{2g}.$$

With respect to this isomorphism, the long exact homotopy sequence of the evaluation fibration (4.1) *becomes*

$$\begin{aligned}\cdots\to\pi_{n+1}(K)\xrightarrow{\delta_{n+1}}\pi_{n+2}(K)\oplus\pi_{n+1}(K)^{2g}\to\pi_n(C(P,K)^K)\\ \to\pi_n(K)\xrightarrow{\delta_n}\pi_{n+1}(K)\oplus\pi_n(K)^{2g}\to\cdots\end{aligned}\tag{4.6}$$

■

The information we have so far on $\pi_n(C(P,K)^K)$ is quite poor, since we have no knowledge on the connecting homomorphisms δ_n yet. We merely get that $C(P,K)^K$ is simply connected in the case of a principal K-bundle over $\mathbb{S}^1$ with simply connected finite-dimensional K. Thus a further treatment of the connecting homomorphisms will be necessary in order to get more crucial information on $\pi_n(C(P,K)^K)$.

Remark 4.1.16. A quite general theorem of Singer [Si78, Theorem 5] states that the weak homotopy type of $C_*(P,K)^K$ is the one of the pointed mapping group

$C_*(M,K)$ if M is a closed manifold of dimension of at most 4 and $K = \mathrm{SU}_n(\mathbb{C})$. The method in the proof is the same that we used in this paragraph. However, our explicit constructions need no assumptions on the homotopy type of K and are aiming for a general treatment of gauge groups with arbitrary structure groups. So the theorem of Singer is of a different flavour. ■

Remark 4.1.17. Similar considerations for the pointed gauge group have been made in [Te05], especially for rational homotopy and rational cohomology. The approach taken there focuses on bundles with simply connected semi-simple structure group over simply connected 4-manifolds and uses the Whitehead-Milnor Theorem to obtain an explicit description of the homotopy type of the base. In combination with [Si78, Theorem 5], the weak homotopy type of the pointed gauge group is reduced to the one of the pointed mapping group on the base, and this result is used to do computations in terms of mapping groups. We are aiming for more general cases and thus take a more general and direct approach by using more explicit constructions. ■

Remark 4.1.18. The explicit description of $\pi_n(C_*(P,K)^K)$ in terms of the homotopy groups of K in Proposition 4.1.11 and Proposition 4.1.15 is the key in our approach to the determination of the homotopy groups of the gauge group $\pi_n(C(P,K)^K)$. As illustrated, this works well for bundles over spheres and compact, closed and orientable surfaces. Furthermore, this approach extends to all classes of bundles for which a good description of $\pi_n(C_*(M,K))$ is available.

Although this will not lead to a systematic understanding of $\pi_n(C(P,K)^K)$ without knowledge on the connecting homomorphisms, the pointed gauge group is an object of its own interest, because is acts freely on the space of connections of $\mathcal{P}$ [MM92, Section 6.4] and thus is often treated as the symmetry group of quantum field theories. Furthermore, a precise knowledge of $\pi_n(C_*(P,K)^K)$ is also desirable, because the non-vanishing of these groups can be seen as a measure for the failure for the existing of global gauges, which is also known as the *Gribov Ambiguity* [MM92, Section 6.5]. ■

Problem 4.1.19. For which manifolds M do we have a description of $\pi_n(C_*(M,K))$ in terms of the homotopy groups of K? ■

4.2 The connecting homomorphisms

This section is devoted to the calculation of the connecting homomorphism in the exact homotopy sequences (4.3) and (4.6) induced by the evaluation fibration. We will not solve this problem in general, but reduce it to a more familiar problem in homotopy theory, i.e., the calculation of Samelson and Whitehead products.

Before starting the calculation of the connecting homomorphisms we give a construction principle for them.

Remark 4.2.1. ([Br93, Theorem VII.6.7]) Let $p : Y \to B$ be a Serre fibration with fibre $F = p^{-1}(\{x_0\})$. The examples of these fibrations that we will encounter in this chapter are locally trivial bundles [Br93, Corollary VII.6.12]. The fibration yields a long exact homotopy sequence

$$\ldots \to \pi_{n+1}(B) \xrightarrow{\delta_{n+1}} \pi_n(F) \xrightarrow{\pi_n(i)} \pi_n(Y) \xrightarrow{\pi_n(q)} \pi_n(B) \xrightarrow{\delta_n} \pi_{n-1}(F) \to \ldots$$

and the construction of the connecting homomorphisms δ_n is as follows: Let $\alpha \in C_*(\mathbb{B}^n, B)$ represent an element of $\pi_n(B)$, i.e., $\alpha|_{\partial\mathbb{B}^n} \equiv x_0$. Then α can be lifted to a base-point preserving map $A : \mathbb{B}^n \to Y$ with $q \circ A = \alpha$, because q is a Serre fibration. Then A takes $\partial\mathbb{B}^n \cong \mathbb{S}^{n-1}$ into $q^{-1}(x_0) = A$, and $A|_{\partial\mathbb{B}^n}$ represents $\delta([\alpha])$. ■

For bundles over compact, closed and orientable surfaces, the connecting homomorphism turns out to be given in terms of the connecting homomorphism for bundles $\mathbb{S}^2$.

Proposition 4.2.2. *Let K be a connected topological group and $\mathcal{P}_{\mathbb{S}^2}$ be a continuous principal K-bundle over $\mathbb{S}^2$, represented by*

$$b \in \pi_1(K) \cong [\mathbb{S}^2, BK]_* \cong \mathrm{Bun}(\mathbb{S}^2, K).$$

(cf. Proposition B.2.8). Denote by $\delta_{n,\mathbb{S}^2} : \pi_n(K) \to \pi_{n+1}(K)$ the n-th connecting homomorphism from the corresponding long exact homotopy sequence for the evaluation fibration (4.3)*. Furthermore, let $\mathcal{P}_\Sigma$ be a continuous principal K-bundle over the compact, closed and orientable surface Σ of genus g, represented by the same*

$$b \in \pi_1(K) \cong [\Sigma, BK]_* \cong \mathrm{Bun}(\Sigma, K)$$

(cf. Proposition B.2.10). Then the n-th connecting homomorphism

$$\delta_{n,\Sigma} : \pi_n(K) \to \pi_{n+1}(K) \oplus \pi_n(K)^{2g}$$

from the long exact homotopy sequence for the corresponding evaluation fibration (4.6) *is given by $\delta_{n,\Sigma}(a) = (\delta_{n,\mathbb{S}^2}(a), 0)$.*

Proof. Let $q : \Sigma \to \mathbb{B}^2$ be the quotient map identifying A_{2g} with the base-point in $\mathbb{S}^2$ (cf. Remark B.2.11). For every principal K-bundle $\mathcal{P}_{\mathbb{S}^2}$ over $\mathbb{S}^2$, we have the corresponding pull-back bundle $\mathcal{P}_\Sigma$ given by

$$\begin{tikzcd} P_\Sigma \arrow[r, "Q"] \arrow[d, "\eta_\Sigma"'] & P_{\mathbb{S}^2} \arrow[d, "\eta_{\mathbb{S}^2}"'] \\ \Sigma \arrow[r, "q"] & \mathbb{S}^2 \end{tikzcd}$$

and $\mathcal{P}_\Sigma$ and $\mathcal{P}_{\mathbb{S}^2}$ have the same representing elements in $\pi_1(K)$(cf. Remark B.2.12). Denote by $\mathrm{ev}_{\mathbb{S}^2} : C(P_{\mathbb{S}^2}, K)^K \to K$ and by $\mathrm{ev}_\Sigma : C(P_\Sigma, K)^K \to K$ the corresponding evaluation fibrations in compatible base-points of P_Σ and $P_{\mathbb{S}^2}$, and observe that $\mathrm{ev}_{\mathbb{S}^2} = \mathrm{ev}_\Sigma \circ Q^*$, where

$$Q^* : C(P_{\mathbb{S}^2}, K)^K \to C(P_\Sigma, K)^K, \quad f \mapsto f \circ Q$$

is the corresponding pull-back. This implies that if $A : \mathbb{B}^n \to C(P_{\mathbb{S}^2}, K)^K$ is a lift of $\alpha : \mathbb{B}^n \to K$ for $\mathrm{ev}_{\mathbb{S}^2}$, then $Q^* \circ A$ is a lift of α for ev_Σ.

Now, let $a \in \pi_n(K)$ be represented by $\alpha : \mathbb{B}^n \to K$ with $\alpha(\partial\mathbb{B}^n) = \{e\}$ and let $A : \mathbb{B}^n \to C(P_{\mathbb{S}^2}, K)^K$ be a lift of α for $\mathrm{ev}_{\mathbb{S}^2}$. Then $Q^* \circ A$ is a lift of α for ev_Σ, and it thus suffices to show that the restriction of the two lifts A and $Q^* \circ A$ to $\partial\mathbb{B}^n$, taking values in $C_*(P_{\mathbb{S}^2}, K)^K$ and $C_*(P_\Sigma, K)^K$, describe the same elements in $\pi_{n+1}(K) \cong \pi_{n+1}(K) \oplus 0$ with respect to the homotopy equivalences in Proposition 4.1.11 and Proposition 4.1.15.

In order to do so, note that a section $\sigma_\Sigma : \Sigma\backslash\{x_0\} \to P_\Sigma$ determines uniquely a section $\sigma_{\mathbb{S}^2} : \mathbb{S}^2\backslash\{x_S\} \to P_{\mathbb{S}^2}$ by $\sigma_{\mathbb{S}^2}(q(x)) = Q(\sigma_\Sigma(x))$, because $q|_{\Sigma\backslash A_{2g}}$ is a homeomorphism onto $\mathbb{S}^2\backslash\{x_S\}$. Thus for each $y \in \partial\mathbb{B}^n$ we have

$$A(y)(Q(\sigma_\Sigma(x))) = A(y)(\sigma_{\mathbb{S}^2}(q(x))),$$

implying

$$Q^*(A(y))(\sigma_\Sigma(x)) = A(y)(\sigma_{\mathbb{S}^2}(q(x))).$$

The homotopy equivalence from Proposition 4.1.11 and 4.1.15 replaces $A|_{\partial\mathbb{B}^n}$ by a mapping with values in $C_{\eta^{-1}(\overline{U})}(P, K)^K$ for some appropriately chosen neighbourhood U of the corresponding base-points. Then the representative of $\delta_{n,\mathbb{S}^2}(a)$ (resp. $\delta_{n,\Sigma}(a)$) is determined by pulling back $A|_{\partial\mathbb{B}^n}$ (resp. $Q^*(A|_{\partial\mathbb{B}^n})$) along $\sigma_{\mathbb{S}^2}$ (resp. along σ_Σ) and extending $\sigma^*_{\mathbb{S}^2}(A(y))$ (resp. $\sigma^*_\Sigma(Q^*(A(y)))$) continuously by e for each $y \in \partial\mathbb{B}^n$ (cf. Proposition 4.1.11 and Proposition 4.1.15). Since $Q^*(A(y))$ vanishes on $\eta_\Sigma^{-1}(q^{-1}(\overline{U}))$, the continuous extension of $\sigma^*_\Sigma(Q^*(A(y)))$ vanishes on $q^{-1}(\overline{U}) \supseteq A_{2g}$.

We eventually see that the $\pi_n(K)$-component of $\delta_{n,\Sigma}(a)$ vanishes. Since the $\pi_{n+1}(K)$-component is determined by identifying elements in $C_*(\mathbb{S}^2, K)$ with $C_{A_{2g}}(\Sigma, K)$ via q, this also yields that the $\pi_{n+1}(K)$-component is $\delta_{\mathbb{S}^2}(a)$. ■

The connecting homomorphism for bundles over spheres will be given in terms of the Samelson product, which we introduce now.

Definition 4.2.3 (Samelson Product). If K is a topological group, $a \in \pi_n(K)$ is represented by $\alpha \in C_*(\mathbb{S}^n, K)$ and $b \in \pi_m(K)$ is represented by $\beta \in C_*(\mathbb{S}^m, K)$, then the commutator map

$$\alpha\#\beta : \mathbb{S}^n \times \mathbb{S}^m \to K, \ \ (x, y) \mapsto \alpha(x)\beta(y)\alpha(x)^{-1}\beta(y)^{-1}$$

maps $\mathbb{S}^n \vee \mathbb{S}^m$ to e. Hence it may be viewed as an element of $C_*(\mathbb{S}^n \wedge \mathbb{S}^m, K)$ and thus determines an element $\langle a,b\rangle_S := [\alpha\#\beta] \in \pi_0(C_*(\mathbb{S}^{n+m},K)) \cong \pi_{n+m}(K)$. Furthermore, it can be shown that $[\alpha\#\beta]$ only depends on the homotopy classes of α and β, and we thus get a map

$$\pi_n(K) \times \pi_m(K) \to \pi_{n+m}(K), \quad (a,b) \mapsto \langle a,b\rangle_S$$

This map is bi-additive [Wh78, Theorem X.5.1] and is called the *Samelson product* (cf. [Wh78, Section X.5]). ■

As indicated before, the connecting homomorphism for bundles over spheres is given in terms of the Samelson product.

Theorem 4.2.4 (Connecting homomorphism is the Samelson product). *If $\mathcal{P}$ is a continuous principal K-bundle over $\mathbb{S}^m$, K is locally contractible and*

$$b \in \pi_{m-1}(K) \cong [\mathbb{S}^m, BK]_* \cong \mathrm{Bun}(\mathbb{S}^m, K)$$

represents $\mathcal{P}$ (cf. Proposition B.2.8), then the connecting homomorphisms

$$\delta_n : \pi_n(K) \to \pi_{n+m-1}(K)$$

in the long exact homotopy sequence

$$\cdots \to \pi_{n+1}(K) \xrightarrow{\delta_{n+1}} \pi_{n+m}(K) \to \pi_n(\mathrm{Gau}_c(\mathcal{P})) \to \pi_n(K) \xrightarrow{\delta_n} \pi_{n+m-1}(K) \to \cdots$$

from Proposition 4.1.11, induced by the evaluation fibration, is given by $\delta_n(a) = -\langle b,a\rangle_S$, where $\langle\cdot,\cdot\rangle_S$ denotes the Samelson product.

Proof. We set $\mathbb{B}^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $\mathbb{S}^n := \{x \in \mathbb{R}^n : \|x\| = 1\}$ and use throughout this proof the identification

$$([0,1] \times \mathbb{S}^{n-1})/(\{0\} \times \mathbb{S}^{n-1}) \xrightarrow{\cong} \mathbb{B}^n, \quad (t,\varphi) \mapsto t\cdot\varphi \tag{4.7}$$

as topological spaces *without base-points*. We denote by $\overline{U}_N$ and $\overline{U}_S$ the *closed* northern and southern hemispheres. Then there are sections $\sigma_N : \overline{U}_N \to P$ and $\sigma_S : \overline{U}_S \to P$ such that the corresponding transition function $k_{\mathcal{P}} : \overline{U}_N \cap \overline{U}_S \cong \mathbb{S}^{m-1} \to K$ represents $\mathcal{P}$ (cf. Remark B.2.9). Since $\overline{U}_N \cong \mathbb{B}^m \cong \overline{U}_S$, we may identify $\mathrm{Gau}_c(\mathcal{P}) \cong C(P,K)^K$ with

$$G_{\overline{\mathcal{U}}}(\mathcal{P}) := \{(f_1,f_2) \in C(\mathbb{B}^m,K)^2 : f_1(x) = k_{\mathcal{P}}(x)\cdot f_2(x)\cdot k_{\mathcal{P}}(x)^{-1} \text{ for all } x \in \partial\mathbb{B}^m\}$$

by the isomorphism $f \mapsto (f\circ\sigma_N, f\circ\sigma_S)$ (cf. Remark 3.2.1). With respect to this identification, the evaluation fibration is given by $\mathrm{ev}(f_1,f_2) = f_2(0)$.

Each $a \in \pi_n(K)$ is represented by $\alpha : [0,1] \times \mathbb{S}^{n-1} \to K$ with $\alpha(\{0,1\} \times \mathbb{S}^{n-1}) = \{e\}$, then we may assume that α even vanishes on

$\{0,1\} \times \mathbb{S}^{n-1} \cup [0,1] \times \{x_0\}$, because $\mathbb{S}^n$ is homotopy equivalent to the reduced suspension $[0,1] \times \mathbb{S}^{n-1}/(\{0,1\} \times \mathbb{S}^{n-1} \cup [0,1] \times \{x_0\})$. We shall construct an explicit lift of α to $G_{\overline{\mathcal{U}}}(\mathcal{P})$.

Since $\overline{U}_N \cong \mathbb{B}^m$ and $\alpha(0) = 0$, we may use the identification (4.7) to define

$$A_N : \mathbb{B}^n \times \overline{U}_N \to K, \quad (x, t \cdot \varphi) \mapsto k(\varphi) \cdot \alpha(t,x) \cdot k(\varphi)^{-1} \tag{4.8}$$

$$A_S : \mathbb{B}^n \times \overline{U}_S \to K, \quad (x,y) \mapsto \alpha(1,x) \tag{4.9}$$

Then $A : \mathbb{B}^n \to C(\mathbb{B}^m, K)^2$, $x \mapsto (A_N(x,\cdot), A_S(x,\cdot))$ defines a continuous map with values in $G_{\mathcal{U}}(\mathcal{P})$, because $t = 1$ if $t \cdot \varphi \in \partial \overline{U}_N$ and thus

$$A_N(x, t\cdot\varphi) = k(\varphi) \cdot \alpha(t,x) \cdot k(\varphi)^{-1} = k(\varphi) \cdot A_S(x, t\cdot\varphi) \cdot k(\varphi)^{-1}.$$

Furthermore, A defines a lift of α, because $\mathrm{ev}(A(x)) = A_2(x,0) = \alpha(x)$.

Since the homotopy equivalence in Proposition 4.1.11 is given by identifying $\overline{U}_S$ with the base-point in $\mathbb{S}^m \cong \mathbb{B}^m/\partial\mathbb{B}^m$ we thus have that $\delta_n(a)$ is given by $[A_N|_{\partial\mathbb{B}^n \times \mathbb{B}^m}]$ in the set of homotopy classes $[\partial\mathbb{B}^n \wedge \mathbb{S}^m, K]_*$. Consider $\widetilde{A} : \mathbb{B}^n \times \mathbb{B}^m \to K$, $(x,y) \mapsto A_N(x,y) \cdot \alpha(x)^{-1}$. Then $\widetilde{A}$

- vanishes on $\partial\mathbb{B}^n \times \partial\mathbb{B}^m$,
- vanishes on $\{x_0\} \times \mathbb{B}^m$, where $x_0 \in \partial\mathbb{B}^n$ is the base-point, because α vanishes on $\{x_0\} \times [0,1]$,
- vanishes on $\mathbb{B}^n \times \{y_0\}$, where $y_0 \in \partial\mathbb{B}^m$ is the base-point, because then $t = 1$ and $\gamma(\varphi) = e$,
- coincides with A_N on $\partial\mathbb{B}^n \times \mathbb{B}^m$, because α vanishes there,
- coincides with $k_{\mathcal{P}}\#\alpha$ on $\mathbb{B}^n \times \partial\mathbb{B}^m$, because then $t = 1$.

We take the coproduct

$$\begin{gathered}\mathbb{S}^{n+m-1} \cong \partial(\mathbb{B}^n \times \mathbb{B}^m) = (\partial\mathbb{B}^n \times \mathbb{B}^m) \cup (\mathbb{B}^n \times \partial\mathbb{B}^m) \to \\ \big((\partial\mathbb{B}^n \times \mathbb{B}^m)/(\partial\mathbb{B}^n \times \partial\mathbb{B}^m \cup \{x_0\} \times \mathbb{B}^m)\big) \cup \big((\mathbb{B}^n \times \partial\mathbb{B}^m)/(\partial\mathbb{B}^n \times \partial\mathbb{B}^m \cup \mathbb{B}^n \times \{y_0\})\big) \\ \cong (\mathbb{S}^{n-1} \wedge \mathbb{S}^m) \cup (\mathbb{S}^n \wedge \mathbb{S}^{m-1}) \to \mathbb{S}^{n+m-1} \vee \mathbb{S}^{n+m-1}\end{gathered}$$

to define the (unique) group structure on $\pi_{n+m-1}(K)$ (cf. [Sp66, Theorem 1.6.8]). We thus have

$$-\langle b, a\rangle_S = -[k_{\mathcal{P}}\#\alpha] = -[\widetilde{A}\big|_{\mathbb{B}^n \times \partial\mathbb{B}^m}] \overset{(*)}{=} [\widetilde{A}\big|_{\partial\mathbb{B}^n \times \mathbb{B}^m}] = \delta_n(a),$$

where $(*)$ follows from [Sp66, Theorem 1.6.8], because $\widetilde{A}$ is a continuous map on $\mathbb{B}^n \times \mathbb{B}^m$ and thus $[\widetilde{A}\big|_{\partial(\mathbb{B}^n \times \mathbb{B}^m)}] = 0$. ∎

As we mentioned before, there is a close interplay between the Samelson and the Whitehead product, which we shall define now.

Definition 4.2.5. Let X be a topological space and $a \in \pi_n(X)$ and $b \in \pi_m(X)$ be represented by $\alpha \in C_{\partial\mathbb{B}^n}(\mathbb{B}^n, X)$ and $\beta \in C_{\partial\mathbb{B}^m}(\mathbb{B}^m, X)$. We identify $\mathbb{S}^{n+m-1}$ with $\partial\mathbb{B}^{n+m} = (\partial\mathbb{B}^n \times \mathbb{B}^m) \cup (\mathbb{B}^m \times \partial\mathbb{B}^n)$ and set

$$(\alpha \diamond \beta) : \mathbb{S}^{n+m-1} \to X, \quad (x,y) \mapsto \begin{cases} \alpha(x) & \text{if } (x,y) \in \mathbb{B}^n \times \partial\mathbb{B}^m \\ \beta(y) & \text{if } (x,y) \in \partial\mathbb{B}^n \times \mathbb{B}^m. \end{cases}$$

Note that this is well-defined, since $\alpha(\partial\mathbb{B}^n) = \{*\} = \beta(\partial\mathbb{B}^m)$. Clearly, the homotopy class of $\alpha \diamond \beta$ depends only on the homotopy classes of α and β and thus determines an element $\langle a, b\rangle_{WH} := [\alpha \diamond \beta] \in \pi_{n+m-1}(X)$, and the map

$$\pi_n(X) \times \pi_m(X) \ni (a,b) \mapsto \langle a, b\rangle_{WH} \in \pi_{n+m-1}(X)$$

is called the *Whitehead product* (cf. [Wh78, Section X.5]). ■

According to [BJS60], the first appearance of the Samelson product seems to be in [Sa53], where it occurs as an explicit formula for the Whitehead product for loop spaces, to make these products more accessible. The general relation between the Samelson and the Whitehead product is the following.

Proposition 4.2.6. *([BJS60, Section 1]) If $\mathcal{P} = (K, \eta : P \to X)$ is a continuous principal K-bundle and $\delta_n : \pi_n(X) \to \pi_{n-1}(K)$ is the n-th connecting homomorphism of the corresponding long exact homotopy sequence for $n \geq 1$, then we have*

$$\delta_{n+m-1}(\langle a, b\rangle_{WH}) = \langle \delta_n(a), \delta_m(m)\rangle_S \tag{4.10}$$

for $a \in \pi_n(X)$ and $b \in \pi_m(X)$ and $n, m \geq 1$. ■

Remark 4.2.7. For a continuous principal K-bundle $\mathcal{P}$ over $\mathbb{S}^m$, the sequence

$$\cdots \to \pi_{n+1}(K) \xrightarrow{\delta_{n+1}} \pi_{n+m}(K) \to \pi_n(C_*(P,K)^K) \to \pi_n(K) \xrightarrow{\delta_n} \pi_{n-1}(K) \to \cdots \tag{4.11}$$

with the connecting homomorphisms from Theorem 4.2.4 can also be obtained as follows. Let $\mathcal{P}_K = (K, \eta_K : EK \to BK)$ be a universal bundle for K, i.e., a continuous principal K-bundle such that $\pi_n(EK)$ vanishes for $n \in \mathbb{N}_0$ (cf. Theorem B.2.4 and Theorem B.2.6). Furthermore, let $\gamma : \mathbb{S}^m \to BK$ be a classifying map for $\mathcal{P}$ and denote by $\Gamma : P \to EK$ the corresponding bundle map, and denote by $C(P, EK)^K$ the space of bundle maps from P to EK.

Now each $f \in C(P, EK)^K$ induces a map $f_{\mathbb{S}^m} : \mathbb{S}^m \to BK$, and the map

$$C(P, EK)^K_\Gamma \ni f \mapsto f_{\mathbb{S}^m} \in C(\mathbb{S}^m, BK)_\gamma \tag{4.12}$$

is a fibration [Go72, Proposition 3.1], where $C(P, EK)^K_\Gamma$ (respectively $C(B, BK)_\gamma$) denotes the connected component of Γ (respectively γ), and we have a homeomorphism

$$F = \{f \in C(P, EK)^K_\Gamma : f_{\mathbb{S}^m} = \gamma\} \cong C(P, K)^K$$

[Go72, Proposition 4.3]. Since $\pi_n(C(P, EK)^K_\Gamma)$ vanishes [Go72, Theorem 5.2], the long exact homotopy sequence of the fibration (4.12) leads to

$$\pi_{n-1}(C(P, K)^K_\Gamma) \cong \pi_n(C(\mathbb{S}^m, BK)_\gamma)$$

(cf. [Ts85, Theorem 1.5] and [AB83, Proposition 2.4]). We now consider the evaluation fibration $\mathrm{ev} : C(\mathbb{S}^m, BK)_\gamma \to BK$, $f \mapsto f(x_S)$. This map is in fact a fibration [Br93, Theorem VII.6.13] with fibre $\mathrm{ev}^{-1}(x_S) =: C_*(\mathbb{S}^m, BK)_\gamma$, and we thus get a long exact homotopy sequence

$$\begin{aligned} \ldots \to \pi_{n+1}(BK) \xrightarrow{\delta_{n+1}} \pi_n(C_*(\mathbb{S}^m, BK)_\gamma) \to \pi_n(C(\mathbb{S}^m, BK)_\gamma) \\ \to \pi_n(BK) \xrightarrow{\delta_n} \pi_{n-1}(C_*(\mathbb{S}^m, BK)_\gamma) \to \ldots . \end{aligned} \tag{4.13}$$

If we identify $\pi_n(C_*(\mathbb{S}^m, BK)_\gamma)$ with $\pi_{n+m}(BK)$ (cf. [Wh46, 2.10]), then the connecting homomorphism in this sequence is given by $\delta_{n+1}(a) = -\langle a, b\rangle_{WH}$, where $b = [\gamma] \in \pi_m(BK)$ and $\langle \cdot, \cdot \rangle_{WH}$ denotes the Whitehead product (cf. [Wh46, Theorem 3.2] and [Wh53, 3.1]).

Since $\pi_n(EK)$ vanishes, each connecting homomorphism $\delta_n : \pi_n(BK) \to \pi_{n-1}(K)$ from the long exact homotopy sequence for $\mathcal{P}_K$ is an isomorphism, and with respect to this identification, the exact sequence from (4.13) becomes (4.11). Since, under this identification, the Whitehead product becomes the Samelson product (cf. (4.10)), the connecting homomorphism is then given by the Samelson product as in Theorem 4.2.4. ■

Remark 4.2.8. For bundles over spheres and over compact closed and orientable surfaces the connecting homomorphisms are given in terms of the Samelson or the Whitehead product. In the case of bundles over surfaces, the reduction of the connecting homomorphisms to the ones for bundles over spheres relies on the fact that these bundles arise from clutching two trivial bundles over a closed 2-cell and the complement of its interior together along a single characteristic map. Now this construction produces more general bundles over more general manifolds (i.e., simply-connected 4-manifolds by the Milnor-Whitehead Theorem, cf. [Te05]), and should lead to more systematic information on $\pi_n(C(P, K)^K)$. ■

Problem 4.2.9. Find more explicit descriptions of principal bundles arising as *simply clutched* bundles, i.e., as bundles over manifolds of dimension n, whose bases possess a trivialising cover consisting of a closed n-cell and the complement of its interior. ■

4.3 Formulae for the homotopy groups

In this section we describe how known results on the Samelson and Whitehead products lead to explicit formulae for the (rational) homotopy groups of the gauge group. This depends on the amount of known results for these products. We are mainly interested in the low-dimensional homotopy groups (i.e., $\pi_1(C(P,K)^K)$ and $\pi_2(C(P,K)^K)$), which causes some problems, because these products are mostly considered in higher dimensions (cf. [Bo60]). However, at least for some examples and in the rational case, these products are well-known.

One example, in which we can use the results of the previous section is the quaterionic Hopf fibration.

Example 4.3.1 (The quaterionic Hopf fibration). Consider the quaternion skew-field $\mathbb{H}\cong\mathbb{R}^4$ with the euclidean norm $\sqrt{q_1\overline{q_1}+\ldots+q_n\overline{q_n}}=\|q\|$ on $\mathbb{H}^n\cong\mathbb{R}^{4n}$ and

$$\mathbb{S}^{4n-1}:=\{q\in\mathbb{H}^n:\|q\|=1\}.$$

Furthermore, consider the projective spaces $\mathbb{PH}^{n-1}:=\mathbb{H}^n/\sim$ with $q\sim q':\Leftrightarrow q=\lambda q'$ for some $\lambda\in\mathbb{S}^3$. Then $\mathbb{S}^3\cong\mathrm{SU}_2(\mathbb{C})$ acts on $\mathbb{S}^{4n-1}$ by $(q_1,\ldots,q_n)\cdot k=(q_1k,\ldots,q_nk)$, and the orbit map composed with the quotient map yields a surjection $\eta':\mathbb{S}^{4n-1}\to\mathbb{PH}^{n-1}$. This defines a continuous principal $\mathrm{SU}_2(\mathbb{C})$-bundle $(\mathrm{SU}_2(\mathbb{C}),\eta':\mathbb{S}^{4n-1}\to\mathbb{PH}^{n-1})$, provided by the trivialisations

$$\eta'^{-1}(U_k)\ni(q_1,\ldots,q_n)\mapsto[(q_1,\ldots,q_n)],|q_k|^{-1}q_k\in U_k\times\mathrm{SU}_2(\mathbb{C}),$$

where $U_k:=\{[q]\in\mathbb{PH}^{n-1}:q_k\neq 0\}$. Now $\mathbb{PH}^1\cong\mathbb{S}^4$, since both spaces are homeomorphic to the one-point compactification of $\mathbb{H}$, and thus we get a continuous principal $\mathrm{SU}_2(\mathbb{C})$-bundle $\mathcal{H}:=(\mathrm{SU}_2(\mathbb{C}),\eta:\mathbb{S}^7\to\mathbb{S}^4)$, called the *quaterionic Hopf fibration*.

A characteristic map (cf. Remark B.2.9) $\gamma:\mathrm{SU}_2(\mathbb{C})\to\mathrm{SU}_2(\mathbb{C})$ for this bundle can obtained as follows. We view $\mathbb{S}^4$ as the quotient of $\{q\in\mathbb{H}:\|q\|\leq 1\}$ by its boundary. Then the homeomorphism from $\mathbb{S}^4$ to the one-point compactification of $\mathbb{H}$ is given by

$$\varphi:\mathbb{S}^4\to\mathbb{H}\cup\{\infty\},\ \ q\mapsto\begin{cases}\frac{q}{1-\|q\|} & \text{if } \|q\|<1\\ \infty & \text{if } \|q\|=1.\end{cases}$$

Composing $\varphi|_{\|q\|<1}$ with the section

$$\sigma':\mathbb{H}\to\mathbb{S}^7,\ \ q\mapsto\frac{1}{\|(1,q)\|}(1,q)$$

yields a map σ, which we may continuously extend to $\|q\|=1$ by setting $\sigma(q)=(0,q)$ in this case. This results in a map σ satisfying $\eta\circ\sigma=q$, where q is the quotient map defining $\mathbb{S}^4$. We thus may take $\gamma=\mathrm{id}_{\mathrm{SU}_2(\mathbb{C})}$ as the map representing the equivalence class of $\mathcal{H}$. ■

More generally, principal $\mathrm{SU}_2(\mathbb{C})$-bundles over $\mathbb{S}^4$ are classified by their so called *Chern number* $k \in \mathbb{Z} \cong \pi_3(\mathrm{SU}_2(\mathbb{C}))$ (cf. [Na97, Theorem 6.4.2]), and the quaterionic Hopf fibration has Chern number 1. We denote by $\mathcal{P}_k$ the principal $\mathrm{SU}_2(\mathbb{C})$-bundle over $\mathbb{S}^4$ with Chern number k.

As mentioned before, the crucial Samelson product in this example is well-known and now leads to an explicit description of $\pi_1(C(P,K)^K)$ and $\pi_2(C(P,K)^K)$ for $\mathrm{SU}_2(\mathbb{C})$-bundles over $\mathbb{S}^4$.

Proposition 4.3.2. *If $\mathcal{P}_k$ is a principal $\mathrm{SU}_2(\mathbb{C})$-bundle over $\mathbb{S}^4$ of Chern number $k \in \mathbb{Z}$, then $\pi_1(C(P_k,K)^K) \cong \mathbb{Z}_2$ and $\pi_2(C(P_k,K)^K) \cong \mathbb{Z}_{\gcd(k,12)}$. In particular, if $\mathcal{P}_1$ denotes the quaterionic Hopf fibration, then $\pi_2(C(P_1,K)^K)$ vanishes.*

Proof. (cf. [Ko91, Lemma 1.3]) Recall the homotopy groups of $\mathrm{SU}_2(\mathbb{C})$ from Remark 4.1.4. First we note that we have $\pi_1(C(P,K)^K) \cong \mathbb{Z}_2$ by the exact sequence

$$\underbrace{\pi_2(\mathrm{SU}_2(\mathbb{C}))}_{=0} \to \underbrace{\pi_5(\mathrm{SU}_2(\mathbb{C}))}_{\cong \mathbb{Z}_2} \to \pi_1(C(P_k,K)^K) \to \underbrace{\pi_1(\mathrm{SU}_2(\mathbb{C}))}_{=0}$$

from Proposition 4.1.11. Since $\mathcal{P}_k$ is classified by the Chern number $k \in \mathbb{Z} \cong \pi_3(\mathrm{SU}_2(\mathbb{C}))$, Theorem 4.2.4 provides an exact sequence

$$\pi_3(\mathrm{SU}_2(\mathbb{C})) \xrightarrow{\delta_2^k} \pi_6(\mathrm{SU}_2(\mathbb{C})) \xrightarrow{\pi_2(i)} \pi_2(C(P_k,K)^K) \to \pi_2(\mathrm{SU}_2(\mathbb{C})),$$

where $\delta_2^k : \pi_3(\mathrm{SU}_2(\mathbb{C})) \to \pi_6(\mathrm{SU}_2(\mathbb{C}))$ is given by $a \mapsto -\langle k,a\rangle_S$. Since $\pi_3(\mathrm{SU}_2(\mathbb{C})) \cong \mathbb{Z}$, $\pi_6(\mathrm{SU}_2(\mathbb{C})) \cong \mathbb{Z}_{12}$ and $\langle 1,1\rangle_S$ generates $\pi_6(\mathrm{SU}_2(\mathbb{C}))$ [Pü04, Corollary 6.2], we may assume that $\delta_2^k : \mathbb{Z} \to \mathbb{Z}_{12}$ is the map $\mathbb{Z} \ni z \mapsto -[kz] \in \mathbb{Z}_{12}$ due to the bi-additivity of $\langle\cdot,\cdot\rangle_S$. Since $\pi_2(\mathrm{SU}_2(\mathbb{C}))$ is trivial, we have that $\pi_2(i)$ is surjective and

$$\mathrm{im}(\pi_2(i)) \cong \mathbb{Z}_{12}/\ker(\pi_2(i)) = \mathbb{Z}_{12}/\mathrm{im}(\delta_2^k) = \mathbb{Z}_{12}/(k\mathbb{Z}_{12}) \cong \mathbb{Z}_{\gcd(k,12)}. \qquad \blacksquare$$

Systematical results on the Samelson product in low dimensions seem not to be available in the literature. This is different for the rational Samelson products, which we will consider now.

Remark 4.3.3. As explained in Section A.2, in infinite-dimensional Lie theory one often considers (period-) homomorphisms $\varphi : \pi_n(G) \to V$ for an infinite-dimensional Lie group G and an $\mathbb{R}$-vector space V, which we consider here as a $\mathbb{Q}$-vector space. If $n \geq 1$, then $\pi_n(G)$ is abelian and this homomorphism factors through the canonical map $\psi : \pi_n(G) \to \pi_n(G) \otimes \mathbb{Q}$, $a \mapsto a \otimes 1$, and

$$\widetilde{\varphi} : \pi_n(G) \otimes \mathbb{Q} \to V, \quad a \otimes x \mapsto x\,\varphi(a).$$

It thus suffices for many interesting questions arising from infinite-dimensional Lie theory to consider the *rational homotopy groups* $\pi_n^{\mathbb{Q}}(G) := \pi_n(G) \otimes \mathbb{Q}$ for $n \geq 1$.

Furthermore, the functor $\otimes\mathbb{Q}$ in the category of abelian groups, sending A to $A^{\mathbb{Q}} := A \otimes \mathbb{Q}$ and $\varphi : A \to B$ to $\varphi^{\mathbb{Q}} := \varphi \otimes \mathrm{id}_{\mathbb{Q}} : A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$, preserves exact sequences, since $\mathbb{Q}$ is torsion free and hence flat (cf. [Br93, Section V.6]). $\blacksquare$

Lemma 4.3.4. *If K is a finite-dimensional Lie group and $n, m \geq 1$, then the rational Samelson product*

$$\langle \cdot, \cdot \rangle_S^{\mathbb{Q}} : \pi_n^{\mathbb{Q}}(K) \times \pi_m^{\mathbb{Q}}(K) \to \pi_{n+m}^{\mathbb{Q}}(K), \quad a \otimes x, b \otimes y \mapsto \langle a, b \rangle_S \otimes xy$$

vanishes.

Proof. We first consider the case where K is connected. If $a \in \pi_n(K)$ and $b \in \pi_m(K)$, then $\langle a, b \rangle_S$ is an element of the torsion subgroup of $\pi_{n+m}(K)$ [Ja59], and the assertion follows from the fact that tensoring with $\mathbb{Q}$ kills the torsion subgroup.

If K is not connected, then $a \in \pi_n(K) \cong \pi_n(K_0)$ is represented by a map $\alpha : \mathbb{S}^n \to K_0$ and $b \in \pi_m(K)$ is represented by a map $\beta : \mathbb{S}^m \to K_0$, because $\mathbb{S}^n$ and $\mathbb{S}^m$ are arcwise connected for $n, m \geq 1$. Then $\alpha \# \beta$ (cf. Definition 4.2.3) also takes values in K_0, as well as $(\alpha \# \beta)^\ell$ for each $\ell \in \mathbb{N}$. Now $\langle [\alpha], [\beta], \rangle_S$ is a torsion element if and only if there exists an integer ℓ_0 such that $(\alpha \# \beta)^{\ell_0}$ is null-homotopic, i.e., extends to $\mathbb{B}^{m+n+1}$. Thus $\ell_0 \langle [\alpha], [\beta] \rangle = 0$, for $[\alpha] \in \pi_n(K_0)$ and $\beta \in \pi_m(K_0)$ if and only if $\ell_0 \langle [\alpha], [\beta] \rangle = 0$, for $[\alpha] \in \pi_n(K)$ and $\beta \in \pi_m(K)$, and the assertion follows from the case where K is connected. ■

Theorem 4.3.5 (Rational homotopy groups of gauge groups). *Let K be a finite-dimensional Lie group and $\mathcal{P}$ be a continuous principal K-bundle over X, and let Σ be a compact orientable surface of genus g. If $X = \mathbb{S}^m$, then*

$$\pi_n^{\mathbb{Q}}(\mathrm{Gau}_c(\mathcal{P})) \cong \pi_{n+m}^{\mathbb{Q}}(K) \oplus \pi_n^{\mathbb{Q}}(K)$$

for $n \geq 1$. If $X = \Sigma$ and K is connected, then

$$\pi_n^{\mathbb{Q}}(\mathrm{Gau}_c(\mathcal{P})) \cong \pi_{n+2}^{\mathbb{Q}}(K) \oplus \pi_{n+1}^{\mathbb{Q}}(K)^{2g} \oplus \pi_n^{\mathbb{Q}}(K)$$

for $n \geq 1$.

Proof. First note, that in the case on a non-closed surface each bundle with connected structure group is trivial (Proposition B.2.10), which yields the assertion in this case. In the other cases, we obtain with Remark 4.3.3 an exact rational homotopy sequence from the exact sequence for the evaluation fibration (4.3) from Proposition 4.1.11 and (4.6) from Proposition 4.1.15. Then the preceding lemma implies that the connecting homomorphisms in these sequences vanish, because the connecting homomorphisms for the homotopy sequences are given in terms of the Samelson product by Proposition 4.2.2 and Theorem 4.2.4. Thus the long exact rational sequence splits into short ones. Furthermore, these short exact sequences split linearly, since each of them involves vector spaces. ■

Remark 4.3.6. The rational homotopy groups of finite-dimensional Lie groups are those of products of odd-dimensional spheres [FHT01, Section 15.f], which are well known [FHT01, Example 15.d.1]. Thus Theorem 4.3.5 gives a detailed description of the rational homotopy groups for the gauge group of bundles over spheres and compact, closed and orientable surfaces.

Although this knowledge is sufficient for many questions in infinite-dimensional Lie theory, it would be desirable to have more explicit descriptions of $\pi_n(C(P,K)^K)$ for larger classes of bundles. As illustrated in Proposition 4.3.2, a detailed knowledge of Samelson- and Whitehead Products would lead to more of these descriptions but this knowledge is not available in low dimensions. ■

Problem 4.3.7. Which explicit formulae for the Samelson- or Whitehead product lead to more explicit descriptions of $\pi_n(C(P,K)^K)$ for larger classes of bundles? ■

Chapter 5

Central extensions of gauge groups

In this chapter we construct a central extension of the identity component $\mathrm{Gau}(\mathcal{P})_0$ of the gauge group and an action of the automorphism group $\mathrm{Aut}(\mathcal{P})$ on it. The procedure is motivated by ideas from [PS86], [LMNS95] and [MN03].

The general idea for constructing central extensions of infinite-dimensional Lie groups is to construct central extensions of the corresponding Lie algebras and then check whether they are induced by corresponding central extensions of their groups. The tools we use here are provided in [Ne02a].

We shall consider bundles over bases without boundary, i.e., our base manifolds will always be *closed* compact manifolds. Throughout this section we fix one particular given smooth principal K-bundle $\mathcal{P}$ over a closed compact manifold M. We furthermore assume K to be locally exponential. This ensures, in particular, that all bundles occurring in this section have the property SUB with respect to each smooth closed trivialising system (cf. Lemma 3.1.13).

5.1 A central extension of the gauge algebra

The first step is to construct central extensions of the gauge algebra. In the case of trivial bundles we have $\mathfrak{gau}(\mathcal{P}) \cong C^\infty(M,\mathfrak{k})$ and from [MN03] the cocycle

$$C^\infty(M,\mathfrak{k}) \times C^\infty(M,\mathfrak{k}) \ni (\eta,\mu) \mapsto [\kappa(\eta, d\mu)] \in \Omega^1(M,Y)/dC^\infty(M,Y), \tag{5.1}$$

where $\kappa : \mathfrak{k} \times \mathfrak{k} \to Y$ is a continuous, symmetric, $\mathfrak{k}$-invariant bilinear map. In this section we shall illustrate how this cocycle generalises to arbitrary smooth bundles by replacing the ordinary differential with a covariant derivative (cf. [LMNS95]).

We first introduce the notation we use throughout this chapter.

Definition 5.1.1. If $\mathcal{P}$ is a smooth principal K-bundle and $\mathrm{Ad}(\mathcal{P})$ is its adjoint bundle, then we have the isomorphisms

$$\mathfrak{gau}(\mathcal{P}) \cong C^\infty(P, \mathfrak{k})^K \cong S(\mathrm{ad}(\mathcal{P})) = \Omega^0(M, \mathrm{ad}(\mathcal{P}))$$

from Proposition 3.1.4. Let Y be a locally convex space, and consider the trivial action $\lambda : K \times Y \to Y$. Then the associated bundle $\lambda(\mathcal{P})$ is trivial, and we thus have $\Omega^1(M, \lambda(\mathcal{P})) \cong \Omega^1(M, Y)$. If $\kappa : \mathfrak{k} \times \mathfrak{k} \to Y$ is a continuous K-invariant bilinear form, then it is in particular K-equivariant with respect to Ad and λ, and we get from Lemma B.3.11 a continuous linear map

$$\kappa_* : \mathfrak{gau}(\mathcal{P}) \times \Omega^1_{\mathrm{bas}}(P, \mathfrak{k})^K \to \Omega^1(M, Y),$$

when identifying $\mathfrak{gau}(\mathcal{P})$ with $\Omega^0(M, \mathrm{Ad}(\mathcal{P}))$ and $\Omega^1_{\mathrm{bas}}(P, \mathfrak{k})^K$ with $\Omega^1(M, \mathrm{Ad}(\mathcal{P}))$ as in Remark B.3.5 and $\Omega^1(M, \lambda(\mathcal{P}))$ with $\Omega^1(M, Y)$. ■

Remark 5.1.2. If M is a closed finite-dimensional manifold and Y is a Fréchet space, then we define

$$\mathfrak{z}_M(Y) := \Omega^1(M, Y)/dC^\infty(M, Y).$$

Since $dC^\infty(M, Y)$ is the annihilator of the continuous linear maps

$$\lambda_\alpha : \Omega^1(M, Y) \to Y, \quad \omega \mapsto \int_{\mathbb{S}^1} \alpha^* \omega, \tag{5.2}$$

for $\alpha \in C^\infty(\mathbb{S}^1, M)$, it follows that $dC^\infty(M, Y)$ is in particular closed in $\Omega^1(M, Y)$ so that we obtain a locally convex Hausdorff vector topology on $\mathfrak{z}_M(Y)$. Furthermore, since $\mathfrak{z}_M(Y)$ is a quotient of the Fréchet space $\Omega^1(M, Y)$ by the closed subspace $dC^\infty(M, Y)$, it is again a Fréchet space. Note that Y is in particular sequentially complete, ensuring the existence of the integral in (5.2). ■

As indicated before, we substitute the ordinary differential in (5.1) by a covariant derivative to obtain the cocycle describing the central extension of $\mathfrak{gau}(\mathcal{P})$.

Lemma 5.1.3. *Let $\mathcal{P}$ be a smooth principal K-bundle over the closed finite-dimensional manifold M, $A \in \Omega^1(P, \mathfrak{k})$ be a connection 1-form and*

$$d^A : \mathfrak{gau}(\mathcal{P}) \to \Omega^1_{\mathrm{bas}}(P, \mathrm{Ad}(\mathcal{P}))^K.$$

be the induced covariant derivative from Lemma B.3.7. If Y is a locally convex space and $\kappa : \mathfrak{k} \times \mathfrak{k} \to Y$ is continuous, bilinear, symmetric and K-invariant, then

$$\omega_{\kappa,A} : \mathfrak{gau}(\mathcal{P}) \times \mathfrak{gau}(\mathcal{P}) \to \mathfrak{z}_M(Y), \quad (\eta, \mu) \mapsto \big[\kappa_*(\eta, d^A \mu)\big] \tag{5.3}$$

is a continuous cocycle on $\mathfrak{gau}(\mathcal{P})$.

Furthermore, if $A, A' \in \Omega^1(P, \mathfrak{k})$ are two connection 1-forms of $\mathcal{P}$, then $\omega_{\kappa,A} - \omega_{\kappa,A'}$ is a coboundary, i.e., there exists a continuous linear map $\lambda : \mathfrak{gau}(\mathcal{P}) \to \mathfrak{z}_M(Y)$ such that we have

$$\omega_{\kappa,A}(\eta, \mu) - \omega_{\kappa,A'}(\eta, \mu) = \lambda([\eta, \mu]) \tag{5.4}$$

for $\eta, \mu \in \mathfrak{gau}(\mathcal{P})$.

Proof. The continuity follows directly from Lemma B.3.11, because $\omega_{\kappa,A}$ is then only a composition of continuous maps. Let $\mathcal{E}_Y$ be the trivial vector bundle $M \times Y$ over M. With the identifications $\Omega^0(M, \mathcal{E}_Y) \cong C^\infty(M, Y)$ and $\Omega^1(M, \mathcal{E}_Y) \cong \Omega^1(M, Y)$, the covariant derivative on $\mathcal{E}_Y$ induced from A is $f \mapsto df$ (cf. Lemma B.3.7).

That $\omega_{\kappa,A}$ is alternating, i.e., $\omega_{\kappa,A}(\eta, \mu) = -\omega_{\kappa,A}(\mu, \eta)$ follows with Lemma B.3.13 from

$$d\kappa_*(\eta, \mu) = \kappa_*(d^A\, \eta, \mu) + \kappa_*(\eta, d^A\, \mu) = \kappa_*(\mu, d^A\, \eta) + \kappa_*(\eta, d^A\, \mu)$$

The cocycle condition is

$$\kappa_*([\eta, \mu], d^A\, \nu) + \kappa_*([\nu, \eta], d^A\, \mu) + \kappa_*([\mu, \nu], d^A\, \eta) \in dC^\infty(M, Y)$$

for all $\eta, \mu, \nu \in \mathfrak{gau}(\mathcal{P})$. With Lemma B.3.13, we get

$$\begin{aligned} d\kappa_*([\eta, \mu], \nu) = \kappa_*(d^A\, [\eta, \mu], \nu) + \kappa_*([\eta, \mu], d^A\, \nu) = \kappa_*([d^A\, \eta, \mu], \nu) + \kappa_*([\eta, d^A\, \mu], \nu) \\ + \kappa_*([\eta, \mu], d^A\, \nu) = \kappa_*([\mu, \nu], d^A\, \eta) + \kappa_*([\nu, \eta], d^A\, \mu) + \kappa_*([\eta, \mu], d^A\, \nu), \end{aligned}$$

because κ is K-invariant and thus $\kappa([x, y], z) = \kappa(x, [y, z])$ for all $x, y, z \in \mathfrak{k}$.

To show that $\omega_{\kappa,A} - \omega_{\kappa,A'}$ is a coboundary, we observe that we get from Lemma B.3.7 $d^A\, \mu - d^{A'}\, \mu = [A' - A, \mu]$, and thus

$$\omega_{\kappa,A}(\eta, \mu) - \omega_{\kappa,A'}(\eta, \mu) = \kappa_*(\eta, [A' - A, \mu]) = \kappa_*(A - A', [\eta, \mu]).$$

Hence $\lambda : \mathfrak{gau}(\mathcal{P}) \to \mathfrak{z}_M(Y)$, $\nu \mapsto [\kappa_*(A - A', \nu)]$ satisfies (5.4). ∎

Definition 5.1.4. The continuous cocycle $\omega_{\kappa,A}$ from the preceding lemma is called *covariant cocycle.*

Remark 5.1.5. Lemma 5.1.3 implies that the class $[\omega_{\kappa,A}] \in H^2_c(\mathfrak{gau}(\mathcal{P}), \mathfrak{z}_M(Y))$ is independent of the choice of the connection 1-form A. Thus, the equivalence class of the central extension

$$\widehat{\mathfrak{gau}(\mathcal{P})}_{\omega_{\kappa,A}} = \mathfrak{z}_M(Y) \oplus_{\omega_{\kappa,A}} \mathfrak{gau}(\mathcal{P}) \text{ with } [(x, \eta), (y, \mu)] = (\omega_{\kappa,A}(\eta, \mu), [\eta, \mu])$$

(cf. Remark A.2.2) does not depend on the choice of A but only on the bundle $\mathcal{P}$ and on κ. ∎

Now the question arises how exhaustive the constructed central extension of $\mathfrak{gau}(\mathcal{P})$ is, i.e., for which spaces it is universal.

Remark 5.1.6. It has been shown in [Ma02] that the central extension of $\mathfrak{gau}(\mathcal{P})$ from Remark 5.1.5 is universal in the case of a trivial bundle, finite-dimensional and semisimple $\mathfrak{k}$ and the universal invariant bilinear form $\mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$, since then $\mathfrak{gau}(\mathcal{P}) \cong C^\infty(M, \mathfrak{k})$ and the cocycle (5.1) is universal.

For non-trivial bundles it is not know to the author whether the central extension of the gauge algebra is universal. The arguments from [Ma02] do not carry over directly, because they use heavily the fact that $\mathfrak{k}$ embeds as a subalgebra into $C^\infty(M, \mathfrak{k})$. This is not true for $C^\infty(P, \mathfrak{k})^K$ and causes the main problem. ■

Problem 5.1.7. For which bundles (beside trivial ones) and for which locally convex spaces is the central extension of Remark 5.1.5 universal? ■

5.2 Integrating the central extension of the gauge algebra

In this and the following section we check the integrability condition for the central extension of $\widehat{\mathfrak{gau}(\mathcal{P})}$ from Remark 5.1.5. The background on central extensions of Lie groups, Lie algebras and their relation is provided in Section A.2.

Unless stated otherwise, throughout this section, we fix one smooth principal K-bundle $\mathcal{P}$ over M for a locally exponential Lie group K and a closed compact manifold M. Furthermore, $\kappa : \mathfrak{k} \times \mathfrak{k} \to Y$ is always a continuous, symmetric and K-invariant bilinear form and a cocycle $\omega_{\kappa,A}$ representing $[\omega_{\kappa,A}]$ from Remark 5.1.5 for an arbitrary connection 1-form A as in Lemma 5.1.3.

Note that we are *not* assuming K to be connected, because this would cause principal bundles over $\mathbb{S}^1$ to become trivial and thus would exclude twisted affine Kac–Moody groups. To this particular class of examples we turn in Section 5.4.

We first motivate the procedure in this section by collecting some results from [Ne02a] and [MN03]. The most important thing that we will have to consider is the period homomorphism associated to a continuous cocycle.

Definition 5.2.1. Let G be a connected Lie group. Then [Ne02a, Section A.3] implies that each class $[\beta] \in \pi_2(G) = \pi_0(C_*(\mathbb{S}^2, G))$ can be represented by a smooth map $\beta \in C^\infty_*(\mathbb{S}^2, G)$. If $\mathfrak{g}$ denotes the Lie algebra of G, $\mathfrak{z}$ is a sequentially complete locally convex (or shortly *s.c.l.c.*) space and $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ is a continuous cocycle, then we define the *period homomorphism*

$$\mathrm{per}_\omega : \pi_2(G) \to \mathfrak{z}, \quad [\beta] \mapsto \int_\beta \Omega,$$

where Ω is the left invariant closed $\mathfrak{z}$-valued 2-form on G with $\Omega(e) = \omega$. Of course, one has to show that this definition does not depend on the choice of the representative β. This is done in [Ne02a, Section 5], where we refer to for the details. There it is also shown that per_ω in fact defines a homomorphism from the abelian group $\pi_2(G)$ into the additive group of $\mathfrak{z}$. ■

The period homomorphism encodes a crucial part of the information on the integrability of the cocycle ω.

Remark 5.2.2. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{z}$ be a s.c.l.c. space. Let $\Gamma \subseteq \mathfrak{z}$ be a discrete subgroup and $Z := \mathfrak{z}/\Gamma$ be the corresponding quotient Lie group. Then we define

$$I : H^2_c(\mathfrak{g}, \mathfrak{z}) \to \mathrm{Hom}(\pi_2(G), Z) \times \mathrm{Hom}(\pi_1(G), \mathrm{Lin}(\mathfrak{g}, \mathfrak{z}))$$

as follows. For the first component we take $I_1([\omega]) := q_Z \circ \mathrm{per}_\omega$, where $q_Z : \mathfrak{z} \to Z$ is the quotient map and $\mathrm{per}_\omega : \pi_2(G) \to \mathfrak{z}$ is the period map of ω. To define $I_2([\omega])$, for each $x \in \mathfrak{g}$, we write X_r for the right invariant vector field on G with $X_r(e) = x$ and Ω for the left invariant $\mathfrak{z}$-valued closed 2-from on G with $\Omega(e) = \omega$. Then $i_{X_r}(\Omega)$ is a closed $\mathfrak{z}$-valued 1-from ([Ne02a, Lemma 3.11]) to which we associate a homomorphism $\pi_1(G) \to \mathfrak{z}$ via

$$I_2([\omega])([\alpha])(x) := \int_\alpha i_{X_r}(\Omega).$$

for a smooth representative $\alpha \in C^\infty_*(\mathbb{S}^1, K)$. We refer to [Ne02a, Section 7] for arguments showing that I is well-defined, i.e., that the right hand side depends only on the cohomology class of ω and the homotopy class of α. ■

Theorem 5.2.3. *([Ne02a, Theorem 7.12]) Let G be a connected Lie group, $\mathfrak{z}$ be a s.c.l.c. space, $\Gamma \subseteq \mathfrak{z}$ be a discrete subgroup and $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ be a continuous Lie algebra cocycle. Then the central extension of Lie algebras $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} := \mathfrak{z} \oplus_\omega \mathfrak{g} \twoheadrightarrow \mathfrak{g}$ integrates, in the sense of Remark A.2.6, to a central extension of Lie groups $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ with $Z_e = \mathfrak{z}/\Gamma$, if and only if $I([\omega]) = 0$.* ■

As we will see later on, the hard part is to check whether I_1 vanishes. By choosing Z appropriately this can always be achieved as long as the image of the period homomorphism is discrete.

Proposition 5.2.4. *Let G be a connected Lie group, $\mathfrak{z}$ be a s.c.l.c. space and $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$. If $\mathrm{per}_\omega : \pi_2(G) \to \mathfrak{z}$ is the associated period homomorphism and the period group $\Pi_\omega := \mathrm{im}(\mathrm{per}_\omega)$ is discrete, then $I_1([\omega])$ from Remark 5.2.2 vanishes if we take $\Gamma = \Pi_\omega$.*

Proof. In this case, $\ker(q_Z) = \mathrm{im}(\mathrm{per}_\omega)$ and thus $I_1([\omega]) = q_Z \circ \mathrm{per}_\omega$ vanishes. ■

In the case that the period group is discrete, one still has to check that I_2 vanishes in order to show that the central extension, determined by ω, integrates. This is always the case if G is simply connected, but in general, the condition that I_2 vanishes seems to be as hard to check as the vanishing of I_1. However, there is an equivalent condition, which makes life easier (at least in the case that we consider here).

Proposition 5.2.5. *([Ne02a, Proposition 7.6]) Let G be a connected Lie group, $\mathfrak{z}$ be a s.c.l.c. space and $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$. Then the adjoint action of $\mathfrak{g}$ on $\mathfrak{z} \oplus_\omega \mathfrak{g}$, given by*

$$(x,(z,y)) \mapsto (\omega(x,y),[x,y]),$$

integrates to a smooth action of G if and only if $I_2([\omega]) = 0$. ■

We now return to our particular cocycle $\omega_{\kappa,A}$. The invariant forms $\kappa : \mathfrak{k} \times \mathfrak{k} \to Y$ that we will mostly work with are the universal ones, which we introduce now.

Definition 5.2.6. If $\mathfrak{k}$ is a locally convex Lie algebra and Y is a locally convex space, then a continuous, symmetric and $\mathfrak{k}$-invariant bilinear form $\kappa : \mathfrak{k}\times\mathfrak{k} \to Y$ is called *universal* if for each $\mathfrak{k}$-invariant symmetric bilinear map $f : \mathfrak{k}\times\mathfrak{k} \to Z$ factors through a unique continuous linear map $\widetilde{f} : Y \to Z$ satisfying $f = \widetilde{f} \circ \kappa$. ■

We collect some facts on universal forms that we use in the sequel. In particular, if $\mathfrak{k}$ is finite-dimensional and simple, then the universal form coincides with the well-known Cartan–Killing form.

Remark 5.2.7. If $\mathfrak{k}$ is finite-dimensional, then a universal, continuous, symmetric $\mathfrak{k}$-invariant bilinear form can be obtained as follows. Denote by $V(\mathfrak{k})$ the quotient $S^2(\mathfrak{k})/\mathfrak{k}.S^2(\mathfrak{k})$, where $S^2(\mathfrak{k})$ is the universal symmetric product, where $\mathfrak{k}$ acts on by $x.(y \vee z) \mapsto [x,y] \vee z + y \vee [x,z]$. Then

$$\kappa : \mathfrak{k}\times\mathfrak{k} \to V(\mathfrak{k}), \quad (x,y) \mapsto [x \vee y],$$

is universal. We shall frequently denote by $\kappa : \mathfrak{k}\times\mathfrak{k} \to V(\mathfrak{k})$ *the* universal form of $\mathfrak{k}$ and consider V as a covariant functor form the category of (finite-dimensional) Lie algebras to (finite-dimensional) vector spaces.

We collect some facts the universal form $\kappa : \mathfrak{k}\times\mathfrak{k} \to V(\mathfrak{k}) \cong \mathbb{R}^n$. The facts used below can be found in the standard literature on (semi-) simple complex and real Lie algebras, e.g., [Ja62], [He78], [Wa01] or [On04]. Note that $n \geq 1$ if $\mathfrak{k}$ is semi-simple, because then the *Cartan–Killing form*

$$\kappa_{CK} : \mathfrak{k}\times\mathfrak{k} \to \mathbb{R}, \quad (x,y) \mapsto \operatorname{tr}(\operatorname{ad}(x)\circ\operatorname{ad}(y))$$

is a symmetric and invariant bilinear form which is non-degenerate by Cartan's Criterion.

Furthermore, since κ_{CK} is non-degenerate, for each other $\mathbb{R}$-valued invariant symmetric bilinear form κ' we find a unique $A \in \mathrm{End}(\mathfrak{k})$ such that $\kappa_{CK}(A.x, y) = \kappa(x, y)$ for all $x, y \in \mathfrak{k}$. Moreover, we have

$$\kappa_{CK}(A.[x,y], z) = \kappa'([x,y], z) = \kappa'(x, [y,z]) = \kappa_{CK}(A.x, [y,z]) = \kappa_{CK}([A.x, y], z)$$

for all $x, y, z \in \mathfrak{k}$, which implies $A.[x,y] = [A.x, y]$. Taking $\mathfrak{k}$ as a module over itself, this implies that A is a module map, i.e., $A \in \mathrm{End}_{\mathfrak{k}}(\mathfrak{k})$. Thus

$$\kappa(x,y) = (\kappa_{CK}(A_1.x, y), \dots \kappa_{CK}(A_n.x, y))$$

for $A_i \in \mathrm{End}_{\mathfrak{k}}(\mathfrak{k})$ and we see that $V(\mathfrak{k}) \cong \mathrm{End}_{\mathfrak{k}}(\mathfrak{k})$ for uniqueness reasons.

If $\mathfrak{k}$ is semi-simple with the simple factors $\mathfrak{k}_1, \dots, \mathfrak{k}_n$, then κ is clearly the direct sum of $\kappa_1, \dots, \kappa_n$, where $\kappa_i : \mathfrak{k}_i \times \mathfrak{k}_i \to V(\mathfrak{k}_i)$ is the universal form of $\mathfrak{k}_i$. This reduces the determination of κ to the case where $\mathfrak{k}$ is simple, so let $\mathfrak{k}$ be a real simple Lie algebra from now on. From the classification of simple real Lie algebras, it follows that $\mathfrak{k}$ is either the restriction of a complex simple Lie algebra to real scalars, or $\mathfrak{k}$ is a real form of a simple complex Lie algebra. In the first case we have that the complexification $\mathfrak{k}^{\mathbb{C}} := \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ is not simple as a complex Lie algebra and in the second case that $\mathfrak{k}^{\mathbb{C}}$ is simple as a complex Lie algebra. We shall treat these cases separately.

If $\mathfrak{k}$ is the restriction of a complex simple Lie algebra to real scalars, then the module maps which are also complex linear, are precisely $\mathrm{End}_{\mathfrak{k}}(\mathfrak{k}) = \mathbb{C} \cdot \mathbb{1}$ by Schur's Lemma. If $\in \mathrm{End}_{\mathfrak{k}}(\mathfrak{k})$ is complex anti-linear, then we deduce from

$$i[x, A.y] = A.(i[x,y]) = -iA.([x,y]) = -i[x, A.y]$$

that it vanishes. By decomposing each $A \in \mathrm{End}_{\mathfrak{k}}(\mathfrak{k})$ in its complex linear and complex anti-linear part we see that this implies $V(\mathfrak{k}) \cong \mathbb{C}$, and the two components of the universal from κ are the real and imaginary part of the Cartan–Killing form of $\mathfrak{k}^{\mathbb{C}}$.

If $\mathfrak{k}^{\mathbb{C}}$ is simple as a complex Lie algebra, then we have $\mathrm{End}_{\mathfrak{k}}(\mathfrak{k}^{\mathbb{C}}) \cong \mathrm{End}_{\mathfrak{k}}(\mathfrak{k}) \otimes_{\mathbb{R}} \mathbb{C}$ and by the same argument as above

$$\mathbb{1} \cdot \mathbb{C} = \mathrm{End}_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{k}^{\mathbb{C}}) \cong \mathrm{End}_{\mathfrak{k}}(\mathfrak{k}^{\mathbb{C}}) \cong (\mathrm{End}_{\mathfrak{k}}(\mathfrak{k})) \otimes_{\mathbb{R}} \mathbb{C},$$

which implies $V(\mathfrak{k}) \cong \mathrm{End}_{\mathfrak{k}}(\mathfrak{k}) \cong \mathbb{R}$. In this case, κ_{CK} is the universal invariant bilinear form. This is particular the case if $\mathfrak{k}$ is a compact Lie algebra, i.e., if κ_{CK} is negative definite or, equivalently, if each Lie group K with $\mathrm{L}(K) = \mathfrak{k}$ is compact. ■

In the case of a finite-dimensional trivial principal K-bundle over $\mathbb{S}^1$ and universal κ, the image of the period homomorphism is known to be discrete. As we will see later on, this is the generic case for all finite-dimensional bundles.

Proposition 5.2.8. *If K is a finite-dimensional Lie group, $\mathcal{P}_K$ is the trivial bundle over $\mathbb{S}^1$ with canonical connection 1-form A, $\kappa : \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ is universal and $\omega_K := \omega_{\kappa,A} \in Z^2_c(\mathfrak{gau}(\mathcal{P}_K), \mathfrak{z}_{\mathbb{S}^1}(Y))$ is the cocycle from Remark 5.1.5, then the associated period group* $\operatorname{im}(\operatorname{per}_{\omega_K}) =: \Pi_{\omega_K}$ *is discrete.*

Proof. We have $\operatorname{Gau}(\mathcal{P}) \cong C^\infty(M,K)$ and $\mathfrak{gau}(\mathcal{P}) \cong C^\infty(M,\mathfrak{k})$, because $\mathcal{P}_K$ is trivial. Then $\operatorname{Ad}(\mathcal{P})$ is also trivial and $f \mapsto df$ is the covariant derivative induced by the canonical connection 1-form on $\mathcal{P}$. Therefore, ω_K coincides with the cocycle $(f,g) \mapsto [\kappa(f,dg)]$ in [MN03, Theorem II.9], where K is assumed to be connected. Since $\pi_2(K) = \pi_2(K_0)$ is trivial,

$$\pi_2(C^\infty(\mathbb{S}^1,K)) \cong \pi_2(C^\infty_*(\mathbb{S}^1,K)) = \pi_2(C^\infty_*(\mathbb{S}^1,K_0)) \cong \pi_2(C^\infty(\mathbb{S}^1,K_0))$$

and $\mathbf{L}(C^\infty(\mathbb{S}^1,K)) \cong C^\infty(\mathbb{S}^1,\mathfrak{k}) \cong \mathbf{L}(C^\infty(\mathbb{S}^1,K_0))$, the image of $\operatorname{per}_{\omega_K}$ is not affected by the missing assumption on K of being connected and [MN03, Theorem II.9] yields the assertion. ■

We now turn to the computation of the image of the period homomorphism in the non-trivial case. As indicated before, bundles over $\mathbb{S}^1$ play a key role in this computation, because we can reduce the situation for arbitrary bundles to the case of bundles over $\mathbb{S}^1$ by choosing appropriate curves $\alpha : \mathbb{S}^1 \to M$ and pull back the bundles along α.

One of the fundamental ideas in bundle theory is that pulling back bundles along homotopic maps does not change the (equivalence class of) the pull-back bundles. We shall adopt this idea and will show that pulling back bundles along homotopic maps $\alpha_1, \alpha_2 : \mathbb{S}^1 \to M$ will not change the (image of) the period homomorphism of the pull-back bundles. This will be the crucial observation to make the whole reduction process to bundles over $\mathbb{S}^1$ work.

Remark 5.2.9. For the entire section we fix a system of representatives $(k_i)_{i\in\pi_0(K)}$ for the group $\pi_0(K) := K/K_0$ of connected components of K with $k_{[e]} = e$. For $\alpha \in C^\infty(\mathbb{S}^1, M)$, we get from Remark B.2.9 that $\alpha^*(\mathcal{P})$ is equivalent to $\mathcal{P}_k$ for some $k \in K$ and that $[k] \in \pi_0(K)$ depends only on the homotopy class of α. We thus obtain a homomorphism $\varphi : \pi_1(M) \to \pi_0(K)$ (which is the connecting homomorphism in the long exact homotopy sequence of $\mathcal{P}$), which satisfies $\alpha^*(\mathcal{P}) \cong \mathcal{P}_{k_{\varphi([\alpha])}}$ and we set $\mathcal{P}_{[\alpha]} := \mathcal{P}_{k_{\varphi([\alpha])}}$. Furthermore, for each $[\alpha] \in \pi_1(M)$ this yields a bundle map $\alpha_{\mathcal{P}} : P_{[\alpha]} \to P$ covering α.

The connection 1-form A on $\mathcal{P}$ induces a connection 1-form A_α on $\mathcal{P}_{[\alpha]}$ by pulling back A to a connection 1-form $\alpha^*_{\mathcal{P}}(A)$ on $\mathcal{P}_{[\alpha]}$. Then the induced covariant derivative d^α satisfies

$$d^\alpha(\eta \circ \alpha_{\mathcal{P}}).X_p = d^A\,\eta.T\alpha_{\mathcal{P}}(X_p) \tag{5.5}$$

for $\eta \in C^\infty(P_{[\alpha]},\mathfrak{k})^K$ and $X_p \in T_pP_{[\alpha]}$. We denote the corresponding cocycle by $\omega_{\kappa,\alpha}$. Furthermore, if α and α' are homotopic, then $\alpha^*_{\mathcal{P}}(A)$ and $\alpha'^*_{\mathcal{P}}(A)$ are two different connection 1-forms on $\mathcal{P}_{[\alpha]}$ and thus $\omega_{\kappa,\alpha} - \omega_{\kappa,\alpha'}$ is a coboundary. Since the period homomorphism of a coboundary vanishes (cf. [Ne02a, Remark 5.9]) and

$$\mathrm{per}_{\omega_{\kappa,\alpha}} - \mathrm{per}_{\omega_{\kappa,\alpha'}} = \mathrm{per}_{\omega_{\kappa,\alpha} - \omega_{\kappa,\alpha'}} = 0,$$

this implies that $\mathrm{per}_{\omega_{\kappa,\alpha}} : \pi_2(\mathrm{Gau}(\mathcal{P}_{[\alpha]})) \to \mathfrak{z}_M(Y)$ depends only on the homotopy class of α and we thus denote it $\mathrm{per}_{\omega_{\kappa,[\alpha]}}$. ■

We now take the mappings between the gauge groups into account that we get from pulling back bundles.

Remark 5.2.10. If $\mathcal{P}$ is a smooth principal K-bundle over the compact manifold M (possibly with corners) and $f : N \to M$ is smooth, then the induced bundle map $f_{\mathcal{P}} : f^*(P) \to P$ induces in turn a map $f_{\mathrm{Gau}} : \mathrm{Gau}(\mathcal{P}) \to \mathrm{Gau}(f^*(\mathcal{P}))$, given by $\gamma \mapsto \gamma \circ f_{\mathcal{P}}$ if we identify $\mathrm{Gau}(\mathcal{P})$ with $C^\infty(P,K)^K$ and $\mathrm{Gau}(f^*(\mathcal{P}))$ with $C^\infty(f^*(P),K)^K$.

Correspondingly, we have a homomorphism $f_{\mathfrak{gau}} : \mathfrak{gau}(\mathcal{P}) \to \mathfrak{gau}(f^*(\mathcal{P}))$, $\eta \mapsto \eta \circ f_{\mathcal{P}}$, which is a morphism of topological Lie algebras by Lemma 2.2.24. It follows that f_{Gau} is a morphism of Lie groups, because $\mathrm{Gau}(\mathcal{P})$ is locally exponential if K is so, and f_{Gau} makes the following diagram commutative

$$\begin{array}{ccc} \mathrm{Gau}(\mathcal{P}) & \xrightarrow{f_{\mathrm{Gau}}} & \mathrm{Gau}(f^*(\mathcal{P})) \\ {\scriptstyle \exp}\uparrow & & {\scriptstyle \exp}\uparrow \\ \mathfrak{gau}(\mathcal{P}) & \xrightarrow{f_{\mathfrak{gau}}} & \mathfrak{gau}(f^*(\mathcal{P})) \end{array}$$

■

We are now able to describe what happens to the period homomorphism when pulling bundles back along smooth curves. The formula derived in the next lemma will be the crucial one to make the reduction to bundles over $\mathbb{S}^1$ work.

Lemma 5.2.11. *If $\alpha \in C^\infty(\mathbb{S}^1, M)$ and λ_α denotes the linear map from Remark 5.1.2, then*

$$\lambda_\alpha \circ \mathrm{per}_{\omega_{\kappa,A}} = \lambda_{\mathrm{id}_{\mathbb{S}^1}} \circ \mathrm{per}_{\omega_{\kappa,[\alpha]}} \circ \pi_2(\alpha_{\mathrm{Gau}}), \tag{5.6}$$

where α_{Gau} is the induced map $\mathrm{Gau}(\mathcal{P}) \to \mathrm{Gau}(\mathcal{P}_{[\alpha]})$ from Remark 5.2.10.

Proof. We identify $\mathrm{Gau}(\mathcal{P})$ and $\mathrm{Gau}(\mathcal{P}_{[\alpha]})$ with $C^\infty(P,K)^K$ and $C^\infty(P_{[\alpha]},K)^K$. Then α_{Gau} is given by $f \mapsto f \circ \alpha_{\mathcal{P}}$, where $\alpha_{\mathcal{P}} : P_{[\alpha]} \to P$ is the induced bundle map.

Denote by $\Omega_{\mathcal{P}}$ and Ω_α the left invariant closed 2-forms on $\mathrm{Gau}(\mathcal{P})$ and $\mathrm{Gau}(\alpha^*(\mathcal{P}))$ with $\Omega_{\mathcal{P}}(e) = \omega_{\kappa,a}$ and $\Omega_\alpha(e) = \omega_{\kappa,\alpha}$. Then $\lambda_\alpha \circ \Omega_{\mathcal{P}}$ is also left invariant, as well as $\alpha^*_{\mathrm{Gau}}(\Omega_\alpha)$ since

$$\begin{aligned} \lambda_\gamma^*(\alpha^*_{\mathrm{Gau}}(\Omega_\alpha)) &= (\alpha_{\mathrm{Gau}} \circ \lambda_\gamma)^*(\Omega_\alpha) \\ &= (\lambda_{\alpha_{\mathrm{Gau}}(\gamma)} \circ \alpha_{\mathrm{Gau}})^*(\Omega_\alpha) = \alpha^*_{\mathrm{Gau}}(\lambda^*_{\alpha_{\mathrm{Gau}}(\gamma)}(\Omega_\alpha)) = \alpha^*_{\mathrm{Gau}}(\Omega_\alpha). \end{aligned}$$

Thus $\alpha_{\mathrm{Gau}}^*(\Omega_\alpha)$ is determined by its values on $T_e\,\mathrm{Gau}(\mathcal{P}) \cong \mathfrak{gau}(\mathcal{P})$, where it is given by

$$(\eta,\mu) \mapsto [\kappa_*(\alpha_{\mathfrak{gau}}(e).\eta, d^A\,\alpha_{\mathfrak{gau}}(e).\mu)] = [\kappa_*(\eta \circ \alpha_{\mathcal{P}}, d^A(\mu \circ \alpha_{\mathcal{P}}))].$$

Since $\lambda_{\mathrm{id}_{\mathbb{S}^1}} \circ \alpha_{\mathrm{Gau}}^*(\Omega_\alpha)$ is also left invariant, we have $\lambda_{\mathbb{S}^1} \circ \alpha_{\mathrm{Gau}}^*(\Omega_\alpha) = \lambda_\alpha \circ \Omega_{\mathcal{P}}$, because

$$\begin{aligned}\int_{\mathbb{S}^1} (\alpha_{\mathrm{Gau}}^*(\Omega_\alpha))(e)(\eta,\nu) &= \int_{\mathbb{S}^1} \kappa_*(\eta \circ \alpha_{\mathcal{P}}, d^\alpha(\mu \circ \alpha_{\mathcal{P}})(\partial_t))dt \\ &\overset{i)}{=} \int_{\mathbb{S}^1} \kappa_*(\eta, d^A\,\mu)(d\alpha(t).\partial_t)dt = \int_\alpha \kappa_*(\eta, d^A\,\mu) = \int_\alpha \Omega_{\mathcal{P}}(e)(\eta,\mu) \quad (5.7)\end{aligned}$$

for $\eta,\mu \in \mathfrak{gau}(\mathcal{P})$, where $i)$ holds due to (5.5). For $\beta \in C^\infty(\mathbb{S}^2, \mathrm{Gau}(\mathcal{P}))$ we thus have

$$\begin{aligned}\lambda_{\mathrm{id}_{\mathbb{S}^1}}\big(\mathrm{per}_{\omega_{\kappa,[\alpha]}}\big(\pi_2(\alpha_{\mathrm{Gau}})([\beta])\big)\big) &= \lambda_{\mathrm{id}_{\mathbb{S}^1}}(\mathrm{per}_{\omega_{\kappa,[\alpha]}}([\alpha_{\mathrm{Gau}} \circ \beta])) \\ = \lambda_{\mathrm{id}_{\mathbb{S}^1}}\Big(\int_{\alpha_{\mathrm{Gau}}\circ\beta} \Omega_\alpha\Big) &= \lambda_{\mathrm{id}_{\mathbb{S}^1}}\Big(\int_\beta \alpha_{\mathrm{Gau}}^*(\Omega_\alpha)\Big) = \int_\beta \lambda_{\mathrm{id}_{\mathbb{S}^1}} \circ \alpha_{\mathrm{Gau}}^*(\Omega_\alpha) \\ &= \int_\beta \lambda_\alpha \circ \Omega_{\mathcal{P}} = \lambda_\alpha\Big(\int_\beta \Omega_{\mathcal{P}}\Big) = \lambda_\alpha(\mathrm{per}_{\mathcal{P},\kappa}([\beta])). \quad \blacksquare\end{aligned}$$

We are now quite close to our aim of showing that pulling back bundles along homotopic maps $\alpha_1, \alpha_2 : \mathbb{S}^1 \to M$ does not change the image of the period group. In view of (5.6), it remains to show that $\pi_2(\alpha_{1,\mathrm{Gau}}) = \pi_2(\alpha_{2,\mathrm{Gau}})$, which follows from the next lemma.

Lemma 5.2.12. *If $\alpha_1, \alpha_2 : \mathbb{S}^1 \to M$ are homotopic, then $\mathcal{P}_{[\alpha_1]} = \mathcal{P}_{[\alpha_2]} =: \mathcal{P}_{[\alpha]}$ and the induced bundle maps $\alpha_{1,\mathcal{P}} : P_{[\alpha]} \to P$ and $\alpha_{2,\mathcal{P}} : P_{[\alpha]} \to P$ are also homotopic.*

Proof. Recall from Remark B.2.9 that a representative $k \in K$ for a bundle over $\mathbb{S}^1$ may be obtained as follows. We identify $\mathbb{S}^1$ with $[0,1]/\{0,1\}$ and denote by $q : [0,1] \to \mathbb{S}^1$ the corresponding quotient map. Then there exists a lift $Q : [0,1] \to P$ with $Q(0) = Q(1) \cdot k$ and k is a representative of the bundle.

Now identify $[0,1] \times \mathbb{S}^1$ with the quotient $[0,1]^2/\sim$ with

$$(x,y) \sim (x',y') :\Leftrightarrow \begin{cases} x = x' & \text{if } y,y' \in \{0,1\} \\ x = x' \text{ and } y = y' & \text{esle} \end{cases}$$

and denote by $q' : [0,1]^2 \to [0,1] \times \mathbb{S}^1$ the corresponding quotient map. Let $F : [0,1] \times \mathbb{S}^1 \to M$ be a homotopy with $F(0,\cdot) = \alpha_1$ and $F(1,\cdot) = \alpha_2$. Then there exists a lift $Q' : [0,1]^2 \to P$ of q', because $[0,1]^2$ is contractible, and we have $Q'(t,0) = Q'(t,1) \cdot k(t)$ for some $k : [0,1] \to K$. Furthermore, $k_t := k(t)$ represents

$F(t,\cdot)^*(\mathcal{P})$ by its construction, i.e., $\mathcal{P}_{k_t} \cong F(t,\cdot)^*(\mathcal{P})$. Finally, k_t depends continuously on t, because

$$k(t) = k_\sigma(Q'(t,1))^{-1} \cdot k_\sigma(Q'(t,0))$$

for an arbitrary section $\sigma : U \to P$ for a trivialising neighbourhood U of $F(t,\{0,1\})$ (cf. Remark B.1.5).

From the identification $\mathcal{P}_{k_t} \cong F(t,\cdot)^*(\mathcal{P})$ we get bundle maps $(F_t)_{\mathcal{P}} : P_{k_t} \to P$. Furthermore, let $R : [0,1]^2 \to K$ be such that $R|_{[0,1]\times\{0\}} \equiv k_{[\alpha]}$ and $R|_{(t,1)} = k_t$. This induces continuous maps $(R_t)_{\mathcal{P}} : P_{k_{[\alpha]}} \to P_{k_t}$ and

$$[0,1] \times P_{k_{[\alpha]}} \to P, \quad (t,p) \mapsto (F_t)_{\mathcal{P}}\big((R_t)_{\mathcal{P}}(p)\big)$$

is a homotopy between $\alpha_{1,\mathcal{P}}$ and $\alpha_{2,\mathcal{P}}$. ■

In order to perform the reduction, we have to know how $\pi_2(f_{\mathrm{Gau}})$ looks in two very special cases.

Lemma 5.2.13. *For $\alpha \in C^\infty(\mathbb{S}^1, M)$, let $\mathcal{P}_{[\alpha]}$ be the bundle over $\mathbb{S}^1$ represented by $k_{\varphi([\alpha])} \in K$ as in Remark 5.2.9. If $f \in C^\infty(\mathbb{S}^1,\mathbb{S}^1)$ is homotopic to the identity, then $\mathcal{P}_{[\alpha]} = \mathcal{P}_{[\alpha\circ f]}$ and*

$$\pi_2(f_{\mathrm{Gau}}) : \pi_2(\mathrm{Gau}(\mathcal{P}_{[\alpha]})) \to \pi_2(\mathrm{Gau}(\mathcal{P}_{[\alpha\circ f]})) = \pi_2(\mathrm{Gau}(P_{[\alpha]}))$$

is the identity map.

On the other hand, if K is finite-dimensional and $f \in C^\infty(\mathbb{S}^1,\mathbb{S}^1)$ is homotopic to a constant map, then $\mathcal{P}_{[\alpha\circ f]} = \mathcal{P}_{[e]}$ is the trivial bundle and

$$\pi_2(f_{\mathrm{Gau}}) : \pi_2(\mathrm{Gau}(\mathcal{P}_{[\alpha]})) \to \pi_2(\mathrm{Gau}(P_{[\alpha\circ f]})) = \pi_2(\mathrm{Gau}(\mathcal{P}_{[e]}))$$

vanishes.

Proof. Lemma 5.2.12 tells us that homotopic maps between the base spaces induce homotopic maps between the gauge groups since they are given by pull-backs of the corresponding bundle maps. If f is homotopic to $\mathrm{id}_{\mathbb{S}^1}$ we may thus assume that $f = \mathrm{id}_{\mathbb{S}^1}$, and then $\pi_2(f_{\mathrm{Gau}})$ is the identity, because f_{Gau} is so. Accordingly, in the case that f is homotopic to the constant map, we may assume that $f \equiv m_0$ and thus $\mathcal{P}_{[\alpha\circ f]} = \mathcal{P}_{[e]}$. In this case $f_{\mathcal{P}}$ has values in one single fibre and thus

$$f_{\mathrm{Gau}} : \mathrm{Gau}(\mathcal{P}_{[\alpha]}) \to \mathrm{Gau}(\mathcal{P}_{[e]}) \cong C^\infty(M,K)$$

takes values in $K \leq C^\infty(M,K)$ and since $\pi_2(K)$ vanishes so does $\pi_2(f_{\mathrm{Gau}})$. ■

One crucial step in the reduction is to show that the image of the period homomorphism is contained in the subspace $H^1_{\mathrm{dR}}(M,Y)$ of $\mathfrak{z}_M(Y)$, which is well accessible.

Remark 5.2.14. Let M be a closed finite-dimensional manifold and Y be a Fréchet space. Since an element $\beta \in \Omega^1(M,Y)$ is an exact form if and only if all integrals $\int_\alpha \beta$ vanish for $\alpha \in C^\infty(\mathbb{S}^1, M)$, the linear maps λ_α separate the points of $\mathfrak{z}_M(Y)$.

A 1-form $\beta \in \Omega^1(M,Y)$ is closed if and only if for all pairs of homotopic paths α, α' we have $\int_\alpha \beta = \int_{\alpha'} \beta$. Therefore, the subspace $H^1_{\mathrm{dR}}(M,Y) \subseteq \Omega^1(M,Y)$ is the annihilator of the linear maps $\lambda_\alpha - \lambda_{\alpha'}$ for $[\alpha] = [\alpha']$ in $\pi_1(M)$. In particular, $H^1_{\mathrm{dR}}(M,Y)$ is a closed subspace of $\mathfrak{z}_M(Y)$. Moreover, we have for $[\beta] \in \mathfrak{z}_M(Y)$ that $[\beta] \in H^1_{\mathrm{dR}}(M,Y)$ if and only if $\lambda_\alpha([\beta])$ only depends on the homotopy class of α.■

We still have to choose our curves $\alpha : \mathbb{S}^1 \to M$ in a way that the image of the period homomorphism of the pull-back bundles carries all information on the image of the period homomorphism on $\mathcal{P}$. This choice is the last thing we have to do before we can prove the Reduction Theorem. This choice makes the space $H^1_{\mathrm{dR}}(M,Y)$ accessible.

Remark 5.2.15. If M is a closed finite-dimensional manifold and Y is a Fréchet space, then the de Rham isomorphism and the Universal Coefficient Theorem (cf. [Br93, Theorem V.7.2]) yield

$$H^1_{\mathrm{dR}}(M,Y) \cong H^1(M,Y) \cong \mathrm{Hom}(H_1(M),Y),$$

because $H_0(M)$ is free. If M is compact, denote by r the rank of the finitely generated free abelian group

$$H_1(M)/\operatorname{tor}(H_1(M))$$

and consider a basis given by the smooth representatives $[\alpha_1], \ldots [\alpha_r]$. Since $H_0(M)$ is free, the Universal Coefficient Theorem and Huber's Theorem (cf. [Hu61] or [Br93, Corollary VII.13.16]) imply

$$\begin{aligned} \mathrm{Hom}(\pi_1(M),\mathbb{Z}) &\cong \mathrm{Hom}(\pi_1(M)/[\pi_1(M),\pi_1(M)],\mathbb{Z}) \\ &\cong \mathrm{Hom}(H_1(M),\mathbb{Z}) \cong H^1(M,\mathbb{Z}) \cong [M,\mathbb{S}^1]. \end{aligned}$$

In particular, there exist maps $f_1, \ldots, f_r \in C^\infty(M,\mathbb{S})$ such that $[f_i \circ \alpha_j] = \delta_{ij} \in \pi_1(\mathbb{S}^1)$, and, in virtue of [Ne02a, Theorem A.3.7], we can assume the f_i to be smooth. Since we chose the α_i to build a basis of $H_1(M)/\operatorname{tor}(H_1(M))$ and each homomorphism from $\operatorname{tor}(H_1(M))$ to Y vanishes, we eventually obtain an isomorphism

$$\Phi : H^1_{\mathrm{dR}}(M,Y) \cong \mathrm{Hom}(H_1(M),Y) \to Y^r, [\beta] \mapsto \Big(\int_{\alpha_i} \beta \Big)_{i=1,\ldots,r}, \tag{5.8}$$

whose inverse is given by $\Phi^{-1}(y_1,\ldots,y_r) \mapsto \sum_{i=1}^r [\delta^l(f_i) \cdot y_i]$. ■

Theorem 5.2.16 (Reduction Theorem). *The period group* $\Pi_{\mathcal{P},\kappa} := \operatorname{im}(\operatorname{per}_{\omega_{\kappa,A}})$ *is contained in the subspace* $H^1_{\mathrm{dR}}(M,Y)$ *of* $\mathfrak{z}_M(Y)$. *If* K *is finite-dimensional,* r *denotes the rank of* $H_1(M)/\operatorname{tor}(H_1(M))$ *and* $\alpha_1,\ldots,\alpha_r \in C^\infty(\mathbb{S}^1,M)$ *and* $f_1,\ldots,f_r \in C^\infty(M,\mathbb{S}^1)$ *are chosen as in Remark 5.2.15, then*

$$\Pi_{\mathcal{P},\kappa} \cong \bigoplus_{i=1}^{r} [\delta^l(f_i)] \cdot \operatorname{im}(\lambda_{\operatorname{id}\mathbb{S}^1} \circ \operatorname{per}_{\omega_{\kappa,[\alpha_i]}}) \cong \bigoplus_{i=1}^{r} \Pi_{\mathcal{P}_{[\alpha_i]},\kappa}. \tag{5.9}$$

In particular, $\Pi_{\mathcal{P},\kappa}$ *is discrete if and only if* $\Pi_{\mathcal{P}_{[\alpha_i]},\kappa}$ *is discrete for* $i=1,\ldots,r$.

Proof. Remark 5.2.9, Lemma 5.2.11 and Lemma 5.2.12 imply that for $\alpha \in C^\infty(\mathbb{S}^1,M)$

$$\lambda_\alpha(\operatorname{per}_{\omega_{\kappa,A}}([\beta])) = \lambda_{\operatorname{id}_{\mathbb{S}^1}}(\operatorname{per}_{\omega_{\kappa,[\alpha]}}([\alpha_{\mathrm{Gau}} \circ \beta])) \in \mathfrak{z}_M(Y)$$

depends only on the homotopy class of α. Consequently, $\operatorname{per}_{\omega_{\kappa,A}}([\beta])$ is an element of $H^1_{\mathrm{dR}}(M,Y)$ by Remark 5.2.14, establishing the first assertion.

In order to show (5.9), we evaluate λ_α on $\operatorname{per}_{\omega_{\kappa,A}}(\pi_2(f_{\mathrm{Gau}}))$ for $\alpha \in C^\infty(\mathbb{S}^1,M)$ and $f \in C^\infty(M,\mathbb{S}^1)$:

$$\begin{aligned}\lambda_\alpha \circ \operatorname{per}_{\omega_{\kappa,A}} \circ \pi_2(f_{\mathrm{Gau}}) = \lambda_{\operatorname{id}_{\mathbb{S}^1}} \circ \operatorname{per}_{\omega_{\kappa,[\alpha]}} \circ \pi_2(\alpha_{\mathrm{Gau}}) \circ \pi_2(f_{\mathrm{Gau}})&\\ = \lambda_{\operatorname{id}_{\mathbb{S}^1}} \circ \operatorname{per}_{\omega_{\kappa,[\alpha]}} \circ \pi_2((\alpha \circ f)_{\mathrm{Gau}}).&\end{aligned}$$

We thus obtain with Lemma 5.2.13

$$\lambda_{\alpha_i} \circ \operatorname{per}_{\omega_{\kappa,A}} \circ \pi_2(f_{j,\mathrm{Gau}}) = \delta_{ij} \cdot \lambda_{\operatorname{id}_{\mathbb{S}^1}} \circ \operatorname{per}_{\omega_{\kappa,[\alpha_i]}}. \tag{5.10}$$

Applying Φ^{-1} to (5.10), we thus obtain

$$\operatorname{im}\big(\operatorname{per}_{\omega_{\kappa,A}} \circ \pi_2(f_{i,\mathrm{Gau}})\big) = [\delta^l(f_i)] \cdot \operatorname{im}(\lambda_{\operatorname{id}_{\mathbb{S}^1}} \circ \operatorname{per}_{\omega_{\kappa,[\alpha_i]}})$$

and hence

$$\Pi_{\mathcal{P},\kappa} \supseteq \bigoplus_{i=1}^{r} [\delta^l(f_i)] \cdot \operatorname{im}(\lambda_{\operatorname{id}_{\mathbb{S}^1}} \circ \operatorname{per}_{\omega_{\kappa,[\alpha_i]}}) \cong \bigoplus_{i=1}^{r} \Pi_{\mathcal{P}_{[\alpha_i]},\kappa}.$$

On the other hand, $\lambda_{\alpha_i} \circ \operatorname{per}_{\omega_{\kappa,A}} = \lambda_{\mathbb{S}^1} \circ \operatorname{per}_{\omega_{\kappa,[\alpha_i]}} \circ \pi_2(\alpha_{i,\mathrm{Gau}})$ implies directly

$$\Pi_{\mathcal{P},\kappa} \subseteq \bigoplus_{i=1}^{r} [\delta^l(f_i)] \cdot \operatorname{im}(\lambda_{\operatorname{id}_{\mathbb{S}^1}} \circ \operatorname{per}_{\omega_{\kappa,[\alpha_i]}}) \cong \bigoplus_{i=1}^{r} \Pi_{\mathcal{P}_{[\alpha_i]},\kappa}.$$

■

In the case of a connected structure group, the pull-back bundles over $\mathbb{S}^1$ are trivial and we thus have the discreteness of the period group that we are aiming for.

Corollary 5.2.17. *If K is finite-dimensional and connected, then the period group $\Pi_{\mathcal{P},\kappa} := \operatorname{im}(\operatorname{per}_{\omega_{\kappa,A}})$ is discrete if and only if $\Pi_{\mathcal{P}_K,\kappa} = \Pi_{\mathbb{S}^1,\kappa}$ is discrete for the trivial bundle $\mathcal{P}_K$ over $\mathbb{S}^1$. Moreover, if $\kappa : \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ is universal, then $\Pi_{\omega_\kappa} := \Pi_{\mathbb{S}^1,\kappa}$ is discrete.*

Proof. Since principal bundles over $\mathbb{S}^1$ are trivial for connected structure groups (cf. Proposition B.2.8), each $\mathcal{P}_{[\alpha_i]}$ in the preceding theorem is in fact trivial and the first assertion follows. Since inner automorphisms induce the identity on $V(\mathfrak{k})$ by its construction, $K = K_0$ acts trivially on $V(\mathfrak{k})$, because it is generated $\exp(\mathfrak{k})$. Thus κ is K-invariant and the second is assertion follows from Proposition 5.2.8. ■

At first glance it does not seem to be a hard restriction to require K to be connected. But since only trivial bundles over $\mathbb{S}^1$ arise in this way, one needs to consider also bundles with non-connected structure groups in order to obtain interesting generalisations of loop groups, e.g., twisted affine Kac–Moody groups (cf. Section 5.4).

Proposition 5.2.18. *If K is finite-dimensional, $k \in K$, and $\mathcal{P}_k$ is the smooth principal K-bundle over $\mathbb{S}^1$ from Remark B.2.9, then the period group $\Pi_{\mathcal{P}_k,\kappa} := \operatorname{im}(\operatorname{per}_{\omega_{\kappa,A_k}})$ equals the period group $\Pi_{\mathbb{S}^1,\kappa} := \operatorname{im}(\operatorname{per}_{\omega_{\kappa,A_e}})$ of the trivial bundle, where A_k and A_e are the canonical connection 1-forms. Furthermore, if $\kappa : \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ is universal and K-invariant, then $\Pi_{\mathcal{P}_k,\kappa}$ is discrete.*

Proof. We identify $\operatorname{Gau}(\mathcal{P}_k)$ with the twisted loop group

$$C^\infty_k(\mathbb{S}^1, K) = \{f \in C^\infty(\mathbb{R}, K) : f(x+1) = k^{-1} \cdot f(x) \cdot k \ \text{ for all } \ x \in \mathbb{R}\},$$

and consider the evaluation fibration $\operatorname{ev}_k : C^\infty_k(\mathbb{S}^1, K) \to K,\ f \mapsto f(0)$. Then we have homotopy equivalences

$$\begin{aligned} \ker(\operatorname{ev}_k) &= \{f \in C^\infty_k(\mathbb{S}^1, K) : f(\mathbb{Z}) = \{e\}\} \\ &\simeq \{f \in C^\infty_k(\mathbb{S}^1, K) : f(\mathbb{Z} + [-\varepsilon, \varepsilon]) = \{e\}\} =: C^\infty_{k,\varepsilon}(\mathbb{S}^1, K) \\ &\cong \{f \in C^\infty_e(\mathbb{S}^1, K) : f(\mathbb{Z} + [-\varepsilon, \varepsilon]) = \{e\}\} =: C^\infty_{e,\varepsilon}(\mathbb{S}^1, K) \\ &\simeq \{f \in C^\infty_e(\mathbb{S}^1, K) : f(\mathbb{Z}) = \{e\}\} = \ker(\operatorname{ev}_e) \end{aligned} \tag{5.11}$$

for $0 < \varepsilon < \frac{1}{2}$. Here the isomorphism $\psi : C^\infty_{k,\varepsilon}(\mathbb{S}^1, K) \to C^\infty_{e,\varepsilon}(\mathbb{S}^1, K)$ is given by first restricting $f \in C^\infty_{k,\varepsilon}(\mathbb{S}^1, K)$ to $[0,1]$ and then extend $f|_{[0,1]}$ to $\widehat{f} : \mathbb{R} \to K$ by defining $\widehat{f}$ to be constant of the $\mathbb{Z}$-translates of $x \in [0,1]$. This implies in particular

$$\left.\widehat{f}\right|_{[0,1]} = f|_{[0,1]} \quad \text{and} \quad \left.\widehat{f}\right|_{\mathbb{Z}+[-\varepsilon,\varepsilon]} = f|_{\mathbb{Z}+[-\varepsilon,\varepsilon]}$$

and thus that $\widehat{f}$ is smooth. Now these homotopy equivalences induce an isomorphism $\Psi : \pi_2(\ker(\operatorname{ev}_k)) \xrightarrow{\cong} \pi_2(\ker(\operatorname{ev}_e))$

Now we have that the inclusions $\iota_k : \ker(\mathrm{ev}_k) \hookrightarrow C^\infty_k(\mathbb{S}^1, K)$ induce surjective maps $\pi_2(\iota_k)$, because $\pi_2(K) = 0$.

We abbreviate $\omega_k := \omega_{\kappa,A_k}$ and $\omega_e := \omega_{\kappa,A_e}$, where A_e is the canonical connection on $\mathcal{P}_e$ and A_k is the canonical connection on $\mathcal{P}_k$ (cf. Lemma B.3.14). We then have the following diagram

$$\begin{tikzcd}
\pi_2(\ker(\mathrm{ev}_k)) \arrow[r, "\pi_2(\iota_k)"] \arrow[d, "\Psi"'] & \pi_2(C^\infty_k(\mathbb{S}^1, K)) \arrow[r, "\mathrm{per}_k"] & \mathfrak{z}_{\mathbb{S}^1}(Y) \arrow[r, "\lambda_{\mathrm{id}_{\mathbb{S}^1}}"] & Y \arrow[d, equal] \\
\pi_2(\ker(\mathrm{ev}_e)) \arrow[r, "\pi_e(\iota_e)"] & \pi_2(C^\infty(\mathbb{S}^1, K)) \arrow[r, "\mathrm{per}_e"] & \mathfrak{z}_{\mathbb{S}^1}(Y) \arrow[r, "\lambda_{\mathrm{id}_{\mathbb{S}^1}}"] & Y,
\end{tikzcd} \tag{5.12}$$

which we claim to be commutative. If $\beta \in C^\infty_*(\mathbb{S}^2, (\ker(\mathrm{ev}_k)))$, then we may assume w.l.o.g. that β takes values in $C^\infty_{k,\varepsilon}(\mathbb{S}^1, K)$, due to the homotopy equivalences (5.11). This implies that the restriction of $\mathrm{per}_k([\beta])$ to $[0,1]$ coincides with the restriction of $\mathrm{per}_e(\Psi([\beta]))$ to $[0,1]$, because $\psi(f)|_{[0,1]} = f|_{[0,1]}$. Since the computation of $\lambda_{\mathrm{id}_{\mathbb{S}^1}}(\mathrm{per}_k([\beta]))$ and $\lambda_{\mathrm{id}_{\mathbb{S}^1}}(\mathrm{per}_e([\beta]))$ involves only the values on $[0,1]$, we deduce that (5.12) is commutative.

Thus $\Pi_{\mathcal{P}_k,\kappa}$ equals $\Pi_{\mathbb{S}^1,\kappa}$, because $\pi_2(\iota_k)$ and $\pi_2(\iota_e)$ are surjective. If we choose $\kappa : \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ to be universal, then $\Pi_{\mathbb{S}^1,\kappa} = \Pi_{\mathcal{P}_k,\kappa}$ is discrete by Proposition 5.2.8. ∎

The following corollary we will need later on when discussing Kac–Moody groups. There we will also encounter examples of interesting forms κ, which are $\mathfrak{k}$-invariant, but not K-invariant and give an outline of possible generalisations.

Corollary 5.2.19. *If K is finite-dimensional, K_0 is compact and $\mathfrak{k}$ is simple over $\mathbb{R}$ and $\kappa : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ is the Cartan–Killing form, then $\lambda_{\mathrm{id}_{\mathbb{S}^1}}(\Pi_{\mathcal{P}_k,\kappa}) \cong \mathbb{Z}$.*

Proof. First note that by Remark 5.2.7 the Cartan–Killing form is universal if $\mathfrak{k}$ is simple and compact. Since $\lambda_{\mathrm{id}^1_{\mathbb{S}}}(\Pi_{\mathcal{P}_k,\kappa}) = \lambda_{\mathrm{id}_{\mathbb{S}^1}}(\Pi_{\mathbb{S}^1,\kappa})$ by Proposition 5.2.18, this follows from the explicit description of $\Pi_{\mathbb{S}^1,\kappa}$ in [MN03, Remark II.10] (where the period group is identified with a subset of Y^r by Φ from (5.8)). ∎

Note that the previous corollary does *not* generalise to simple $\mathfrak{k}$, because then the universal form need *not* be K-invariant (cf. Example 5.4.13).

5.3 Actions of the automorphism group

In this section we will construct a smooth action of the automorphism group $\mathrm{Aut}(\mathcal{P})$ on the central extension $\widehat{\mathrm{Gau}(\mathcal{P})}_0$ from Theorem 5.3.8. This will in particular finish integration of the central extension of $\mathfrak{gau}(\mathcal{P})$, which we began in the previous chapter.

Unless stated otherwise, throughout this section we fix a finite-dimensional smooth principal K-bundle $\mathcal{P}$ over a closed compact manifold M. Throughout this section we will assume the bundles to be finite-dimensional, because this makes severals smoothness arguments for actions easier.

In order to make things not too complicated, we assume our bundles throughout this sections to be finite-dimensional.

We start with the construction of various actions of $\operatorname{Aut}(\mathcal{P})$.

Remark 5.3.1. If $\mathcal{P}$ is a principal K-bundle and $\lambda : K \times Y \to Y$ is a smooth action, then we have a canonical action of $\operatorname{Aut}(\mathcal{P})$ on $C^\infty(P,Y)^\lambda$, given by

$$\operatorname{Aut}(\mathcal{P}) \times C^\infty(P,Y)^\lambda \to C^\infty(P,Y)^\lambda, \quad F.\eta = \eta \circ F^{-1}. \tag{5.13}$$

Furthermore, each $F \in \operatorname{Aut}(\mathcal{P})$ induces a diffeomorphism F_M on M and thus $\operatorname{Aut}(\mathcal{P})$ acts on $C^\infty(M,Y)$ and $\Omega^1(M,Y)$ by

$$\operatorname{Aut}(\mathcal{P}) \times C^\infty(M,Y) \to C^\infty(M,Y), \quad F.\eta = \eta \circ F_M^{-1} = (F_M^{-1})^*\eta$$

and

$$\operatorname{Aut}(\mathcal{P}) \times \Omega^1(M,Y) \to \Omega^1(M,Y), \quad F.\omega = \omega \circ TF_M^{-1} = (F_M^{-1})^*\omega.$$

Furthermore, these actions are smooth, because $F \mapsto F_M$ is smooth and $\operatorname{Diff}(M)$ acts smoothly on $C^\infty(M,Y)$ and $\Omega^1(M,Y)$ by Lemma 2.2.25 and Lemma 2.2.26. Since this action preserves the subspace $dC^\infty(M,Y) \subseteq \Omega^1(M,Y)$, it factors through a smooth action

$$\operatorname{Aut}(\mathcal{P}) \times \mathfrak{z}_M(Y) \to \mathfrak{z}_M(Y), \quad F.[\omega] = [F.\omega]. \tag{5.14}$$

■

Lemma 5.3.2. *If $\lambda = \operatorname{Ad}$ is the adjoint action, then the action* (5.13) *of* $\operatorname{Aut}(\mathcal{P})$ *on $C^\infty(P,\mathfrak{k})^K$ is smooth and automorphic.*

Proof. In view of Remark 3.4.8, this is simply the adjoint action of $\operatorname{Aut}(\mathcal{P})$, restricted to the ideal $\operatorname{L}(\operatorname{Gau}(\mathcal{P})) \cong C^\infty(P,\mathfrak{k})^K$, which is smooth and automorphic. ■

We now collect several properties of the pull-back action of $\operatorname{Aut}(\mathcal{P})$ on $\Omega^1(P,\mathfrak{k})$. This action will be the one that relates the actions of $\operatorname{Aut}(\mathcal{P})$ on $\mathfrak{gau}(\mathcal{P})$ and on $\mathfrak{z}_M(Y)$ to give an action of $\operatorname{Aut}(\mathcal{P})$ on the central extension $\mathfrak{z}_M(Y) \oplus_{\omega_{\kappa,A}} \mathfrak{gau}(\mathcal{P})$. In other words, this action will yield a cocycle for the action on this central extension (cf. Remark A.3.5).

Remark 5.3.3. If $\mathcal{P}$ is a smooth principal K-bundle, $A \in \Omega^1(P,\mathfrak{k})$ is a connection 1-form and $F \in \operatorname{Aut}(\mathcal{P})$, then $F^*A := A \circ TF$ is also a connection 1-form. In fact, we have

$$\rho_{F(p)}(k) = F(p)\cdot k = F(p\cdot k) = F \circ \rho_p(k),$$
$$\tau_{F(p)}(x) = T\rho_{F(p)}(e).x = TF \circ T\rho_p(e).x = TF(\tau_p(x)),$$

thus

$$A \circ TF \circ T\rho_k = A \circ T(\rho_k \circ F) = \mathrm{Ad}(k^{-1}).(A \circ TF),$$
$$A(TF(\tau_p(x))) = A(\tau_{F(p)}(x)) = x$$

and F^*A is again a connection 1-from. This gives us an action

$$\mathrm{Aut}(\mathcal{P}) \times \mathrm{Conn}(\mathcal{P}) \to \mathrm{Conn}(\mathcal{P}), \quad F.A = (F^{-1})^*A$$

of $\mathrm{Aut}(\mathcal{P})$ on the affine space $\mathrm{Conn}(\mathcal{P})$ of connection 1-forms on $\mathcal{P}$. ■

Lemma 5.3.4. *If $\mathcal{P}$ is a finite-dimensional smooth principal K-bundle over the closed compact manifold M, then the action*

$$r : \mathrm{Aut}(\mathcal{P}) \times \Omega^1(P, \mathfrak{k}) \to \Omega^1(P, \mathfrak{k}), \quad F \mapsto A - (F^{-1})^*A,$$

is smooth.

Proof. As in Proposition 3.4.15 it can bee seen that the canonical action $\mathrm{Aut}(P) \times TP \to TP$, $F.X_p = TF(X_p)$ is smooth. Since P is finite-dimensional and the topology on $\Omega^1(P, \mathfrak{k})$ is the induced topology from $C^\infty(TP, \mathfrak{k})$, the assertion now follows from Lemma 2.2.25. ■

We shall only need a special case of the previous lemma, where we fix a connection 1-from A and then let $\mathrm{Aut}(\mathcal{P})$ act on A.

Remark 5.3.5. Let $\mathcal{P}$ be a principal K-bundle, A be a connection 1-from on $\mathcal{P}$ and $F \in \mathrm{Aut}(\mathcal{P})$. Then F^*A is again a connection 1-from and the difference $A - F^*A$ vanishes on each vertical tangent space V_p, because each $X_p \in V_p$ can be written as $\tau_p(x)$ for $x \in \mathfrak{k}$ and we have

$$A(X_p) - A(TF(X_p)) = A(\tau_p(x)) - A(TF(\tau_p(x))) = x - x = 0.$$

Thus $A - F^*A \in \Omega^1_{\mathrm{bas}}(P, \mathfrak{k})^K \cong \Omega^1(M, \mathrm{Ad}(\mathcal{P}))$ and we get a map

$$r_A : \mathrm{Aut}(\mathcal{P}) \to \Omega^1(M, \mathrm{Ad}(\mathcal{P})), \quad F \mapsto A - (F^{-1})^*A. \tag{5.15}$$

Furthermore, r_A is a 1-cocycle, i.e., we have $r_A(F \cdot F') = r_A(F) + F.r_A(F')$. Here, the action of $\mathrm{Aut}(\mathcal{P})$ on $\Omega^1(M, \mathrm{Ad}(\mathcal{P}))$ is given by the canonical action

$$\mathrm{Aut}(\mathcal{P}) \times \Omega^1(P, \mathfrak{k})^K \to \Omega^1(P, \mathfrak{k})^K, \quad (F, A) \mapsto (F^{-1})^*A$$

which leaves the subspace $\Omega^1_{\mathrm{bas}}(P, \mathfrak{k})^K$ invariant, and is compatible with the isomorphism $\Omega^1_{\mathrm{bas}}(P, \mathfrak{k})^K \cong \Omega^1(M, \mathrm{Ad}(\mathcal{P}))$. ■

Lemma 5.3.6. *If $A \in \Omega^1(P,\mathfrak{k})^K$ is a connection 1-form, then the cocycle*

$$r_A : \mathrm{Aut}(\mathcal{P}) \to \Omega^1_{\mathrm{bas}}(P,\mathfrak{k})^K \cong \Omega^1(M,\mathrm{Ad}(\mathcal{P})), \quad F \mapsto A - (F^{-1})^* A$$

is smooth. Furthermore, for $\eta \in \mathfrak{gau}(\mathcal{P})$ we have $dr_A(e).\eta = -\,d^A\,\eta$.

Proof. We only have to show $dr_A(e).\eta = -\,d^A\,\eta$. In order to do so, we first derive a formula for $A - (F^{-1})^* A$. Identifying $\mathrm{Gau}(\mathcal{P})$ with $C^\infty(P,K)^K$ by $\gamma \mapsto F_\gamma = \rho \circ (\mathrm{id}_P \times \gamma) \circ \Delta$ (cf. Remark 3.1.2) we have for $X_p \in T_p P$

$$\begin{aligned}
(F_\gamma^{-1})^* A(X_p) &= A \circ TF\gamma^{-1}(X_p) \\
&= A \circ T\rho(X_p, T\gamma^{-1}(X_p)) \\
&= A \circ T\rho(X_p, 0_{\gamma^{-1}(p)}) && + A \circ T\rho(0_p, T\gamma^{-1}(X_p)) \\
&= A \circ T\rho_{\gamma^{-1}(p)}(X_p) && + A \circ T\rho_p(T\gamma^{-1}(X_p)) \\
&= \mathrm{Ad}(\gamma(p))(A(X_p)) && + A \circ T\rho_{p\cdot\gamma^{-1}(p)}(\delta^l(\gamma^{-1})(X_p)) \\
&= \mathrm{Ad}(\gamma(p))(A(X_p)) && + A \circ \tau_{p\cdot\gamma^{-1}(p)}(\delta^l(\gamma^{-1})(X_p)) \\
&= \mathrm{Ad}(\gamma(p))(A(X_p)) && + \delta^l(\gamma^{-1})(X_p).
\end{aligned}$$

This yields the well-known transformation formula for connections (cf. [Na00, Section 1.4])

$$(F_\gamma^{-1})^* A = \mathrm{Ad}(\gamma).A + \delta^l(\gamma^{-1}) = \mathrm{Ad}(\gamma).A + \gamma.d\gamma^{-1}.$$

Now [MN03, Lemma III.2] shows that $d(\delta^l)(e).\eta = d\eta$ (cf. also [GN07a]) and we thus obtain

$$\begin{aligned}
(dr_A(e).\eta)(X_p) &= \left.\frac{d}{dt}\right|_{t=0} r_A(e)(\exp(t\cdot\eta))(X_p) = \left.\frac{d}{dt}\right|_{t=0} \big(r_A(e)(\exp(t\cdot\eta)(X_p)\big) \\
&= \left.\frac{d}{dt}\right|_{t=0} \big(\mathrm{Ad}(\exp(t\cdot\eta(p)), A(X_p)) + \delta^l(\exp(t\cdot\eta)^{-1})(X_p)\big) \\
&= \big(\mathrm{ad}(A(X_p),\eta(p))\big) - d\eta(X_p) = -\,d^A\,\eta(X_p).
\end{aligned}$$ ∎

As we said before, the cocylce r_A now yields a cocycle for an action of $\mathrm{Aut}(\mathcal{P})$ on the central extension $\mathfrak{z}_M(Y) \oplus_\omega \mathfrak{gau}(\mathcal{P})$.

Proposition 5.3.7. *Let $\mathcal{P}$ be a finite-dimensional smooth principal K-bundle over the closed compact manifold M and A be a connection 1-form on $\mathcal{P}$. If Y is a Fréchet space, $\kappa : \mathfrak{k} \times \mathfrak{k} \to Y$ is K-invariant, symmetric, bilinear and continuous and $\omega_{\kappa,A}$ is the continuous cocycle from Lemma 5.1.5, then $\mathrm{Aut}(\mathcal{P})$ acts smoothly and automorphically on $\widehat{\mathfrak{gau}(\mathcal{P})} = \mathfrak{z}_M(Y) \oplus_{\omega_{\kappa,A}} \mathfrak{gau}(\mathcal{P})$ by*

$$F.(z,\eta) = (F.z + R_A(F,\eta), F.\eta), \tag{5.16}$$

where F acts on $\mathfrak{gau}(\mathcal{P}) \cong C^\infty(P,\mathfrak{k})^K$ by (5.13), on $\mathfrak{z}_Y(M)$ by (5.14) and

$$R_A : \mathrm{Aut}(\mathcal{P}) \times \mathfrak{gau}(\mathcal{P}) \to \mathfrak{z}_M(Y), \quad (F,\eta) \mapsto [\kappa_*(F.\eta, r_A(F))].$$

Proof. First we check that (5.16) in fact defines an action of abstract groups. Since r_A is a 1-cocycle, we have

$$\begin{aligned}
&F'.(F.(z,\eta)) \\
=&F'.(F.z+R_A(F,\eta),F.\eta) \\
=&((F'\cdot F).z+F'.R_A(F,\eta)+R_A(F',F.\eta),(F'\cdot F).\eta) \\
=&((F'\cdot F).z,(F'\cdot F).\eta) \\
&\quad+\Big(F'.[\kappa_*(F.\eta,r_A(F))]+\big[\kappa_*\big((F'\cdot F).\eta,r_A(F')\big)\big],(F'\cdot F).\eta\Big) \\
=&((F'\cdot F).z,(F'\cdot F).\eta) \\
&\quad+\Big([\kappa_*(F'\cdot F.\eta,F'.r_A(F))]+\big[\kappa_*\big((F'\cdot F).\eta,r_A(F')\big)\big],(F'\cdot F).\eta\Big) \\
=&\big((F'\cdot F).z+\big[\kappa_*((F'\cdot F).\eta,r_A(F'\cdot F))\big],(F'\cdot F).\eta\big) \\
=&\big((F'\cdot F).z+R(F'\cdot F,\eta),(F'\cdot F).\eta)\big) \\
=&(F'\cdot F).(z,\eta).
\end{aligned}$$

That $\operatorname{Aut}(\mathcal{P})$ acts by Lie algebra automorphisms follows from the description of automorphisms of central extensions in Lemma A.2.3, because we have

$$\begin{aligned}
R_A(F'\cdot F,\eta)&=[\kappa_*(F'\cdot F.\eta,r_A(F'\cdot F))]=[\kappa_*(F'\cdot F.\eta,r_A(F')+F'.r_A(F))]\\
&=[\kappa_*(F'\cdot F.\eta,r_A(F'))]+F'.[\kappa_*(F.\eta,r_A(F))]=R(F',F.\eta)+F'.R_A(F,\eta)
\end{aligned}$$

Finally, the action is smooth because $\operatorname{Aut}(\mathcal{P})$ acts smoothly on $\mathfrak{z}_Y(M)$ and $\mathfrak{gau}(\mathcal{P})$ and because r_A is smooth. ■

We are now ready to prove the two main results of this chapter.

Theorem 5.3.8 (Integrating the the central extension of $\mathfrak{gau}(\mathcal{P})$). *Let $\mathcal{P}$ be a finite-dimensional smooth principal K-bundle over the closed compact manifold M and $\kappa:\mathfrak{k}\times\mathfrak{k}\to V(\mathfrak{k})$ be universal. Furthermore, set $\mathfrak{z}:=\mathfrak{z}_M(V(\mathfrak{k}))$, $\mathfrak{g}:=\mathfrak{gau}(\mathcal{P})$ and $G:=\operatorname{Gau}(\mathcal{P})_0$. If A is a connection 1-form on $\mathcal{P}$, d^A its covariant derivative and*

$$\omega:=\omega_{\kappa,A}:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{z},\quad(\eta,\mu)\mapsto[\kappa_*(\eta,d^A\mu)]$$

is the cocycle from Lemma B.3.11, then $I([\omega])=0$, where

$$I:H^2_c(\mathfrak{g},\mathfrak{z})\to\operatorname{Hom}(\pi_2(G),\mathfrak{z}/\Pi_\omega)\times\operatorname{Hom}(\pi_1(G),\operatorname{Lin}(\mathfrak{g},\mathfrak{z})).$$

is the map from Remark 5.2.2 and Π_ω is the period group $\Pi_\omega=\operatorname{im}(\operatorname{per}_\omega)$ of ω. Thus the central extension

$$\mathfrak{z}\hookrightarrow\widehat{\mathfrak{g}}_\omega\twoheadrightarrow\mathfrak{g}\tag{5.17}$$

of Lie algebras integrates to an extension of Lie groups

$$Z\hookrightarrow\widehat{G}\twoheadrightarrow G,$$

with $Z_0=\mathfrak{z}/\Pi_\omega$.

Proof. First we note that Π_ω is discrete by Theorem 5.2.16 and Proposition 5.2.18, and thus $\mathfrak{z}/\Pi_\omega$ is in fact a Lie group. Since the central extension (5.17) integrates if and only if $I([\omega])$ vanishes (cf. Theorem 5.2.3) and $I_1([\omega])$ vanishes by its construction, have to check that $I_2([\omega])=0$.

Recall that we defined $R_A:\mathrm{Aut}(\mathcal{P})\times\mathfrak{g}\to\mathfrak{z},\ (F,\eta)\mapsto[\kappa_*(F.\eta,r_A(F))]$. Restricting the action

$$\mathrm{Aut}(\mathcal{P})\times\mathfrak{z}\oplus_\omega\mathfrak{g}\to\mathfrak{z}\oplus_\omega\mathfrak{g},\quad F.(z,\eta)=(F.z+R_A(F,\eta),F.\eta)$$

from (5.16) to G, we get a smooth action λ of G on $\widehat{\mathfrak{g}}_\omega$ by $F.(z,\eta)=(R_A(F,\eta),F.\eta)$, because $F_M=\mathrm{id}_M$ if $F\in\mathrm{Gau}(\mathcal{P})$. We calculate the derived action of $\mathfrak{g}$. First observe that for $\eta,\mu\in\mathfrak{g}$ we have

$$\begin{aligned}dR_A(e,\mu).(\eta,0)&=[d\kappa_*(\mu,0)(\mathrm{ad}(\eta,\mu),dr_A(e).\eta)]\\&=[\kappa_*(\mu,dr_A(e).\eta)]+[d\kappa_*(0,\mathrm{ad})]=[\kappa_*(\mu,-\,d^A.\eta)],\end{aligned}$$

since $dr_A(e).\eta=-\,d^A\eta$ by Lemma 5.3.6. Thus

$$\begin{aligned}\dot{\lambda}(\eta).(z,\mu)=d\lambda(e,(z,\mu)).(\eta,(0,0))&=(dR_A(e,\mu).\eta,d\,\mathrm{Ad}(e,\mu).(\eta,0))\\&=([\kappa_*(\mu,-\,d^A\eta)],[\eta,\mu])=(\omega(\eta,\mu),[\eta,\mu])\end{aligned}$$

implies that the derived action of $\mathfrak{g}$ on $\widehat{\mathfrak{g}}_\omega$ is the adjoint action of $\mathfrak{g}$ on $\widehat{\mathfrak{g}}$. By Proposition 5.2.5, this is the case if and only if $I_2([\omega])$ vanishes. This establishes the assertion. ■

Theorem 5.3.9 (Integrating the $\mathrm{Aut}(\mathcal{P})$-action on $\widehat{\mathfrak{gau}(\mathcal{P})}$). *Let $\mathcal{P}$ be a finite-dimensional smooth principal K-bundle over the closed compact manifold M and $\kappa:\mathfrak{k}\times\mathfrak{k}\to Y$ be continuous, bilinear and K-invariant. If $\omega_{\kappa,A}$ is the cocycle from Lemma B.3.11, the period group $\Pi_{\mathcal{P},\kappa}:=\mathrm{im}(\mathrm{per}_{\omega_{\kappa,A}})$ is discrete and*

$$Z\hookrightarrow\widehat{\mathrm{Gau}(\mathcal{P})}_0\twoheadrightarrow\mathrm{Gau}(\mathcal{P})_0$$

is a central extension of $\mathrm{Gau}(\mathcal{P})_0$ *with $Z_0=\mathfrak{z}_M(Y)/\Pi_{\mathcal{P},\kappa}$, then the smooth action of* $\mathrm{Aut}(\mathcal{P})$ *on $\widehat{\mathfrak{gau}(\mathcal{P})}$ from Proposition 5.3.7 integrates to a smooth action of* $\mathrm{Aut}(\mathcal{P})$ *on* $\widehat{\mathrm{Gau}(\mathcal{P})}_0$.

Proof. We abbreviate $G=\mathrm{Gau}(\mathcal{P})_0$. The construction of $\widehat{G}$ in [Ne02a, Lemma 7.11] shows that we have

$$\begin{array}{ccccc}Z_0 & \longrightarrow & H & \xrightarrow{q_H} & \widetilde{G}\\ \downarrow & & {\scriptstyle q'}\downarrow & & {\scriptstyle q_G}\downarrow\\ Z_0 & \longrightarrow & \widehat{G} & \xrightarrow{q} & G,\end{array}$$

where H is the central extension of the universal covering group $\widetilde{G}$ determined by ω (note that $\mathrm{L}(\widetilde{G}) = \mathrm{L}(G) = \mathfrak{gau}(\mathcal{P})$) and $\widehat{G} \cong H/E$ for a discrete subgroup $E \cong \pi_1(G)$ of H.

Using [MN03, Lemma V.5], we lift the conjugation of $\mathrm{Aut}(\mathcal{P})$ on G to a smooth action of $\mathrm{Aut}(\mathcal{P})$ on $\widetilde{G}$, having the same induced action on $\mathfrak{gau}(\mathcal{P})$. Furthermore, the action of $\mathrm{Aut}(\mathcal{P})$ on $\mathfrak{z}_M(Y)$ preserves Π_ω and thus $\mathrm{Aut}(\mathcal{P})$ acts also on Z_0, inducing the canonical action on $\mathfrak{z}_M(Y)$. Then the Lifting Theorem [MN03, Theorem V.9] yields the assertion. ∎

As in the end of Section 5.1, the question arises how exhaustive the constructed central extension of $\mathrm{Gau}(\mathcal{P})_0$ is, i.e., for which spaces it is universal. Furthermore, one would like to know whether this central extension can be enlarged to a central extension of the whole gauge group $\mathrm{Gau}(\mathcal{P})$.

Remark 5.3.10. In [MN03, Section IV] it is shown that the central extension $\widehat{\mathrm{Gau}(\mathcal{P})}_0$ from Theorem 5.2.3 is universal for a large class of groups in the case of a trivial bundle (where $\mathrm{Gau}(\mathcal{P}) \cong C^\infty(M, K)$) and finite-dimensional and semisimple $\mathfrak{k}$. The proof given there would carry over to show universality of $\widehat{\mathrm{Gau}(\mathcal{P})}_0$ as well, if we knew that the central extension $\widehat{\mathfrak{gau}(\mathcal{P})}$ was universal. We thus see once more the importance of Problem 5.1.7.

The question whether the central extension of $\mathrm{Gau}(\mathcal{P})_0$ can be enlarged to a central extension of $\mathrm{Gau}(\mathcal{P})$ has not been considered so far. ∎

Problem 5.3.11. We abbreviate $\mathrm{Gau}(\mathcal{P}) := G$. When does the central extension

$$Z \hookrightarrow \widehat{G_0} \xrightarrow{q_0} G_0$$

from Theorem 5.3.8 extend to a central extension of G, i.e., when does there exist a central extension

$$Z \hookrightarrow G \xrightarrow{q} G$$

and a homomorphism $\varphi : \widehat{G_0} \to \widehat{G}$ such that the diagram

$$\begin{array}{ccccc} Z & \longrightarrow & \widehat{G_0} & \xrightarrow{q_0} & G_0 \\ \| & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \iota} \\ Z & \longrightarrow & \widehat{G} & \xrightarrow{q} & G \end{array}$$

commutes? ∎

5.4 Kac–Moody groups

In this section we describe the relation of gauge groups to (affine, topological) Kac–Moody groups. As indicated in the beginning of Section 5.2, these groups arise as central extensions of gauge groups for bundles over $\mathbb{S}^1$, where the twisted affine Kac–Moody groups arise as gauge groups for non-trivial bundles, i.e., for non-connected structure group (cf. Proposition B.2.8).

Trivial bundles form one particular equivalence class of bundles. From this point of view, generalisations of affine Kac–Moody groups are at hand, e.g., by considering (central extensions) of gauge groups over flat bundles or by considering more general structure groups (cf. Remark 5.4.14). We thus see bundle theory as the natural framework for a unified treatment of Kac–Moody groups and their various generalisations.

Since there are many different flavours of Kac-Moody groups we first fix our setting.

Definition 5.4.1. If K is a Lie group, then for $k \in K$ we define the *twisted loop group*

$$C_k^\infty(\mathbb{S}^1, K) := \{\gamma \in C^\infty(\mathbb{R}, K) : \gamma(x+n) = k^{-n} \cdot \gamma(x) \cdot k^n \text{ for all } x \in \mathbb{R}, n \in \mathbb{Z}\}.$$

and the *twisted loop algebra*

$$C_k^\infty(\mathbb{S}^1, \mathfrak{k}) := \{\eta \in C^\infty(\mathbb{R}, \mathfrak{k}) : \eta(x+n) = \operatorname{Ad}(k)^{-n}.\eta(x) \text{ for all } x \in \mathbb{R}, n \in \mathbb{Z}\}.$$

■

Lemma 5.4.2. *Let K be a flat principal K-bundle over M, given by*

$$P_\varphi = \widetilde{M} \times K/\sim \;\; \text{with } (\widetilde{m}, k) \sim (\widetilde{m} \cdot d, \varphi(d)^{-1} \cdot k)$$

for a homomorphism $\varphi : \pi_1(M) \to K$ (cf. Remark B.3.15). Then

$$\operatorname{Gau}(\mathcal{P}) \cong C^\infty(\widetilde{M}, K)^{\pi_1(M)} := \{f \in C^\infty(\widetilde{M}, K) : f(\widetilde{m} \cdot d) = \varphi(d)^{-1} \cdot f(\widetilde{m}) \cdot \varphi(d)\}$$

and

$$\mathfrak{gau}(\mathcal{P}) \cong C^\infty(\widetilde{M}, \mathfrak{k})^{\pi_1(M)} := \{f \in C^\infty(\widetilde{M}, \mathfrak{k}) : f(\widetilde{m} \cdot d) = \operatorname{Ad}(\varphi(d))^{-1}.f(\widetilde{m})\}.$$

In particular, if $\mathcal{P}_k$ is a principal K-bundle over $\mathbb{S}^1$, given by some $k \in K$ (cf. Remark B.2.9), then $\operatorname{Gau}(\mathcal{P}_k) \cong C_k^\infty(\mathbb{S}^1, K)$ *and* $\mathfrak{gau}(\mathcal{P}_k) \cong C_k^\infty(\mathbb{S}^1, \mathfrak{k})$.

Proof. The isomorphism for that gauge group is provided by

$$C^\infty(\widetilde{M}, K)^{\pi_1(M)} \ni f \mapsto \big([(\widetilde{m}, k)] \mapsto k^{-1} \cdot f(\widetilde{m}) \cdot k\big) \in C^\infty(P_\varphi, K)^K.$$

That the map on the right-hand-side is well-defined follows from the $\pi_1(M)$-equivariance of f and that it is K-equivariant follows directly from the definition of the K-action on P_φ. The isomorphism for the gauge algebra is given by

$$C^\infty(\widetilde{M}, \mathfrak{k})^{\pi_1(M)} \ni f \mapsto \big([(\widetilde{m}, k)] \mapsto \operatorname{Ad}(k)^{-1}.f(\widetilde{m})\big) \in C^\infty(P_\varphi, K)^K.$$

■

Remark 5.4.3. Note that $C^{\infty}_{k}(\mathbb{S}^1,K)$ is isomorphic to the loop group $C^{\infty}(\mathbb{S}^1,K)$ if $k\in K_0$. In fact, then we can find a curve $\tau\in C^{\infty}(\mathbb{R},K)$ satisfying $\tau(x+n)=\tau(x)\cdot k^n$ for $x\in\mathbb{R}$, $n\in\mathbb{Z}$ and then $\gamma\mapsto\tau\cdot\gamma\cdot\tau^{-1}$ provides such an isomorphism. Thus we recover the fact from the classification of bundles over $\mathbb{S}^1$, that they are classified up to equivalence by $\pi_0(K)$. ■

We now endow $C^{\infty}_{k}(\mathbb{S}^1,K)$ with a topology turning the above isomorphism into isomorphism of topological groups.

Remark 5.4.4. We endow $C^{\infty}_{k}(\mathbb{S}^1,K)$ with the subspace topology from the C^{∞}-topology on $C^{\infty}(\mathbb{R},K)$ and the construction in Lemma 5.4.2 shows that it is also isomorphic to $\operatorname{Gau}(\mathcal{P}_k)$ as a topological group. Consequently, it is a Lie group modelled on $C^{\infty}_{k}(\mathbb{S}^1,\mathfrak{k})$. ■

In order to make our definition of a Kac–Moody group precise, we first collect some material on central extensions of twisted loop algebras and groups.

Remark 5.4.5. Let K be a (not necessarily connected) finite-dimensional Lie group such that $\mathfrak{k}$ is a compact real simple Lie algebra. If $\kappa:\mathfrak{k}\times\mathfrak{k}\to\mathbb{R}$ is the Cartan–Killing form, then κ is in particular K-invariant, since $\kappa(x,y)=\operatorname{tr}(\operatorname{ad}(x)\circ\operatorname{ad}(y))$ is invariant under $\operatorname{Aut}(\mathfrak{k})$. It furthermore is universal (cf. Remark 5.2.7).

If $\mathcal{P}_k$ is a smooth principal K-bundle over $\mathbb{S}^1$, then we have a canonical connection 1-form on it inducing the covariant derivative $f\mapsto df$, if we identify $\mathfrak{gau}(\mathcal{P})$ with $C^{\infty}_{k}(\mathbb{S}^1,\mathfrak{k})$ (cf. Lemma B.3.14). We thus have a canonical cocycle

$$\omega: C^{\infty}_{k}(\mathbb{S}^1,\mathfrak{k})\times C^{\infty}_{k}(\mathbb{S}^1,\mathfrak{k})\to\mathfrak{z}_{\mathbb{S}^1}(\mathbb{R})\cong\mathbb{R},\quad(\eta,\mu)\mapsto\int_{[0,1]}\kappa(\eta,\mu')$$

if we identify $\mathfrak{z}_{\mathbb{S}^1}(\mathbb{R})$ with $\mathbb{R}$ as in Remark 5.2.14. This defines a central extension

$$\mathbb{R}\hookrightarrow\mathbb{R}\oplus_{\omega}C^{\infty}_{k}(\mathbb{S}^1,\mathfrak{k})\twoheadrightarrow C^{\infty}_{k}(\mathbb{S}^1,\mathfrak{k}),\tag{5.18}$$

which is equivalent to the central extension of $\mathfrak{gau}(\mathcal{P}_k)$ by $\mathfrak{z}_{\mathbb{S}^1}(\mathbb{R})$ from Remark 5.1.5. Furthermore, by Theorem 5.2.3 this central extension integrates to a central extension

$$Z\hookrightarrow \widehat{C^{\infty}_{k}(\mathbb{S}^1,K)}_0\twoheadrightarrow C^{\infty}_{k}(\mathbb{S}^1,K)_0\tag{5.19}$$

of $C^{\infty}_{k}(\mathbb{S}^1,K)_0$ with $Z_0\cong\mathbb{R}/\Pi_{\omega}$, where $\Pi_{\omega}=\operatorname{im}(\operatorname{per}_{\omega})$ is the image of the period homomorphism $\operatorname{per}_{\omega}:\pi_2(C^{\infty}_{k}(\mathbb{S}^1,K))\to\mathfrak{z}_{\mathbb{S}^1}(\mathbb{R})\cong\mathbb{R}$. We assume from now on that K_0 is simply connected. Then the exact sequence

$$\pi_1(K)\to\pi_0(C^{\infty}_{k}(\mathbb{S}^1,K))\xrightarrow{\pi_0(\mathrm{ev})}\pi_0(K)$$

from the evaluation fibration shows that $C^{\infty}_{k}(\mathbb{S}^1,K)$ maps injectively into $\pi_0(K)$. Since $\mathbb{S}^1$ is connected, the image of $\pi_0(\mathrm{ev})$ are precisely the components K^k of K

mapped onto themselves by conjugation with k, i.e., $\mathrm{im}(\pi_0(\mathrm{ev})) = \mathrm{Fix}_{\pi_0(K)}([k])$. Thus we have $C^\infty_k(\mathbb{S}^1, K) = C^\infty_k(\mathbb{S}^1, K) \cap C^\infty(\mathbb{S}^1, K^k)$. Furthermore, the exact sequence

$$\pi_2(K) \to \pi_1(C^\infty_k(\mathbb{S}^1, K)) \to \pi_1(K)$$

from the evaluation fibration shows that $\pi_1(C^\infty_k(\mathbb{S}^1, K))$ vanishes, because $\pi_1(K) = \pi_1(K_0)$ and $\pi_2(K) = \pi_2(K_0)$. Thus the exact sequence

$$\pi_1(C^\infty_k(\mathbb{S}^1, K)) \to \pi_0(Z) \to \pi_0(C^\infty_k(\mathbb{S}^1, K))$$

from the long exact homotopy sequence of the locally trivial bundle (5.19) shows that Z is also connected. Furthermore, if K_0 is compact, which is equivalent to $\mathfrak{k}$ being the compact real from of a simple complex Lie algebra (cf. [He78, Proposition X.1.5] and [DK00, Corollary 3.6.3]), then Corollary 5.2.19 shows that $\Pi_\omega \cong \mathbb{Z}$ and we thus have in fact a central extension

$$\mathbb{T} \hookrightarrow \widehat{C^\infty_k(\mathbb{S}^1, K)} \twoheadrightarrow C^\infty_k(\mathbb{S}^1, K), \tag{5.20}$$

which is unique (up to equivalence), because $\pi_1(C^\infty_k(\mathbb{S}^1, K))$ is simply connected. ■

The following definition seems implicitly to be contained in the literature, but the author was not able to find a precise reference for it. One reference often used is [PS86], but there the meaning of a Kac–Moody group in the twisted case (i.e., the case of non-connected K) is not made precise.

According to the algebraic definition of a Kac–Moody group (cf. [PK83]), it should be a group which "integrates" the central extension (5.18). Thus the following definition seems to be appropriate.

Definition 5.4.6. If K is a finite-dimensional Lie group with simple real Lie algebra $\mathfrak{k}$, then we call the central extension $\mathfrak{g}_k := \widehat{C^\infty_k(\mathbb{S}^1, \mathfrak{k})}$ from (5.18) an *affine Kac–Moody algebra*. If, moreover, K_0 is compact and simply-connected, then the central extension $G_k := \widehat{C^\infty_k(\mathbb{S}^1, K)}$ from (5.20) is called an *affine Kac–Moody group*. ■

In the compact case, $\mathfrak{g}_k$ and G_k can be seen as unitary real forms of complex Kac–Moody algebras and groups.

Remark 5.4.7. Note that the equivalence class of the central extensions $\mathfrak{g}_k$ and G_k only depends on $[k] \in \pi_0(K)$, because the equivalence class of the bundle $\mathcal{P}_k$ does so and equivalent bundles lead to equivalent extensions. ■

Let's see which topological information on G_k we have.

Proposition 5.4.8. *For the affine Kac-Moody group G_k and the twisted loop group $C^\infty_k(\mathbb{S}^1, K)$ we have that $\pi_1(G_k)$, $\pi_1(C^\infty_k(\mathbb{S}^1, K))$ and $\pi_2(G_k)$ vanish and $\pi_2(C^\infty_k(\mathbb{S}^1, K)) \cong \mathbb{Z}$. For $n \geq 3$ we have $\pi_n(G_k) \cong \pi_n(C^\infty_k(\mathbb{S}^1, K))$.*

Proof. Since $\mathbb{T}$ is a $K(1,\mathbb{Z})$ (i.e., $\pi_n(\mathbb{T})$ vanishes except for $n=1$ and $\pi_1(\mathbb{T})\cong\mathbb{Z}$), the long exact homotopy sequence of the locally trivial fibration 5.20 immediately yields the cases $n\geq 3$ and furthermore leads to

$$\underbrace{\pi_2(\mathbb{T})}_{=0}\to\pi_2(G_k)\to\pi_2(C^\infty_k(\mathbb{S}^1,K)_0)\xrightarrow{\delta_1}\underbrace{\pi_1(\mathbb{T})}_{\cong\mathbb{Z}}\to\pi_1(G_k)\to\pi_1(C^\infty_k(\mathbb{S}^1,K)_0)\to\underbrace{\pi_0(\mathbb{T})}_{=1}. \tag{5.21}$$

Since the connecting homomorphism δ_1 is precisely $-\operatorname{per}_\omega$ [Ne02a, Proposition 5.11], it is in particular surjective, because $\mathbb{T}\cong\mathbb{R}/\operatorname{im}(\operatorname{per}_\omega)$. From the exact sequence

$$\underbrace{\pi_3(K)}_{\cong\mathbb{Z}}\to\pi_2(C^\infty_k(\mathbb{S}^1,K))\to\underbrace{\pi_2(K)}_{=0}\to\underbrace{\pi_2(K)}_{=0}\to\pi_1(C^\infty_k(\mathbb{S}^1,K))\to\underbrace{\pi_1(K)}_{=0} \tag{5.22}$$

induced by the evaluation fibration, we get immediately that $\pi_1(C^\infty_k(\mathbb{S}^1,K))$ vanishes. This implies in turn that $\pi_1(G_k)$ vanishes, because $\delta_1=-\operatorname{per}_\omega$ is surjective and thus (5.21) implies that $\pi_1(G_k)$ maps invectively into $\pi_1(C^\infty_k(\mathbb{S}^1,K))$. Thus $\pi_1(G_k)\cong\pi_1(C^\infty_k(\mathbb{S}^1,K))=0$.

Furthermore, (5.22) implies that $\pi_2(C^\infty_k(\mathbb{S}^1,K))$ is a quotient of $\pi_3(K)\cong\mathbb{Z}$ and hence cyclic. Since δ_1 is surjective, $\pi_2(C^\infty_k(\mathbb{S}^1,K))$ must be infinite and thus is isomorphic to $\mathbb{Z}$. Since δ_1 is surjective, (5.21) now implies $\pi_2(G_k)=0$. ■

Note that $\pi_1(G_k)=0$ justifies the the terminology "affine Kac–Moody group", because it allows continuous representations of $\mathfrak{g}_k$ to be lifted to smooth actions of G_k, at least in the case of continuous representations on Banach spaces (cf. [PK83] and [Ne06b, Theorem IV.1.19.]).

Often, Kac–Moody algebras are introduced as central extensions of twisted loop algebras, given in terms of finite order automorphisms of $\mathfrak{k}$. This we relate now to our notion of twisted loop algebra.

Remark 5.4.9. If $\mathfrak{k}$ is a finite-dimensional simple real Lie algebra and $\varphi\in\operatorname{Aut}(\mathfrak{k})$ is of finite order r, then we set

$$C^\infty_\varphi(\mathbb{S}^1,\mathfrak{k}):=\{\eta\in C^\infty(R,\mathfrak{k}):\eta(x+n)=\varphi^n(\eta(x))\}.$$

If φ is an inner automorphism, then the twisted loop algebra is isomorphic to the untwisted loop algebra, since then φ can be connected with $\operatorname{id}_\mathfrak{k}$ in $\operatorname{Aut}(\mathfrak{k})$ by a smooth path, which yields an isomorphism (cf. Remark 5.4.3). We will thus assume from now on that φ is an outer automorphism.

In this situation, if $\widetilde{K}$ is a finite-dimensional simply connected Lie group with Lie algebra $\mathfrak{k}$, then φ integrates to a uniquely determined automorphism $\Phi:\widetilde{K}\to\widetilde{K}$, which has also order r. Then $\mathbb{Z}_r$ acts on $\widetilde{K}$ by $[m].g=\Phi^m(g)$ and we set $K:=\mathbb{Z}_r\ltimes_\Phi\widetilde{K}$. Then the Lie algebra of K is also $\mathfrak{k}$, and unwinding the definitions we get $\operatorname{Ad}(1,e)=\varphi$. Furthermore, K is non-connected, because $\pi_0(K)\cong\mathbb{Z}_r$.

Thus $(1,e)$ determines a non-trivial principal K-bundle $\mathcal{P}_\varphi := \mathcal{P}_{(1,e)}$ over $\mathbb{S}^1$ and we have
$$C^\infty_{(1,e)}(\mathbb{S}^1,\mathfrak{k}) \cong C^\infty_\varphi(\mathbb{S}^1,\mathfrak{k}). \qquad \blacksquare$$

After having related the constructed central extension of gauge groups to affine Kac–Moody groups, we turn to an application of the construction of the Lie group structure on $\operatorname{Aut}(\mathcal{P}_k)$, which turns out to be the automorphism group of $C^\infty_k(\mathbb{S}^1,\mathfrak{k})$.

Example 5.4.10 ($\operatorname{Aut}(C^\infty_k(\mathbb{S}^1,\mathfrak{k}))$). Let K be a finite-dimensional Lie group, K_0 be compact and simply connected and $\mathcal{P}_k$ be a smooth principal K-bundle over $\mathbb{S}^1$. From Lemma 5.3.2 we get a smooth action of $\operatorname{Aut}(\mathcal{P}_k)$ on $C^\infty_k(\mathbb{S}^1,\mathfrak{k})$, which also lifts to an action on $\widehat{C^\infty_k(\mathbb{S}^1,\mathfrak{k})}$. Various results (cf. [Le80, Theorem 16]) assert that each automorphism of $C^\infty_k(\mathbb{S}^1,\mathfrak{k})$ arises in this way and we thus have a geometric description of $\operatorname{Aut}(C^\infty_k(\mathbb{S}^1,\mathfrak{k}))\cong\operatorname{Aut}(\mathcal{P}_k)$. Furthermore, this also leads to topological information on $\operatorname{Aut}(C^\infty_k(\mathbb{S}^1,\mathfrak{k}))$, since we get a long exact homotopy sequence

$$\begin{aligned} \ldots \to \pi_{n+1}(\operatorname{Diff}(\mathbb{S}^1)) \xrightarrow{\delta_{n+1}} \pi_n(C^\infty_k(\mathbb{S}^1,K)) \to \pi_n(\operatorname{Aut}(\mathcal{P}_k)) \\ \to \pi_n(\operatorname{Diff}(\mathbb{S}^1)) \xrightarrow{\delta_n} \pi_{n-1}(C^\infty_k(\mathbb{S}^1,K)) \to \ldots \end{aligned} \tag{5.23}$$

induced by the locally trivial bundle $\operatorname{Gau}(\mathcal{P}_k)\hookrightarrow\operatorname{Aut}(\mathcal{P}_k)\xrightarrow{q}\!\!\!\!\to\operatorname{Diff}(\mathbb{S}^1)_{\mathcal{P}_k}$ from Theorem 5.3.9 and the isomorphisms $\operatorname{Gau}(\mathcal{P}_k)\cong C^\infty_k(\mathbb{S}^1,K)$ and $\operatorname{Aut}(\mathcal{P}_k)\cong\operatorname{Aut}(C^\infty_k(\mathbb{S}^1,\mathfrak{k}))$. E.g., in combination with

$$\pi_n(\operatorname{Diff}(\mathbb{S}^1)) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n=0 \\ \mathbb{Z} & \text{if } n=1 \\ 0 & \text{if } n\geq 2 \end{cases} \tag{5.24}$$

(cf. [Mi84]), one obtains information on $\pi_n(\operatorname{Aut}(\mathcal{P}_k))$. In fact, consider the exact sequence

$$\begin{aligned} 0 \to \pi_1(C^\infty_k(\mathbb{S}^1,K)) \to \pi_1(\operatorname{Aut}(\mathcal{P}_k)) \to \underbrace{\pi_1(\operatorname{Diff}(M))}_{\cong\mathbb{Z}} \to \pi_0(C^\infty_k(\mathbb{S}^1,K)) \\ \to \pi_0(\operatorname{Aut}(\mathcal{P}_k)) \xrightarrow{\pi_0(q)} \pi_0(\operatorname{Diff}(\mathbb{S}^1)_{\mathcal{P}_k}) \end{aligned}$$

induced by (5.23) and (5.24). Since $\pi_1(C^\infty_k(\mathbb{S}^1,K))$ vanishes, this implies $\pi_1(\operatorname{Aut}(\mathcal{P}_k))\cong\mathbb{Z}$. A generator of $\pi_1(\operatorname{Diff}(\mathbb{S}^1))$ is $\operatorname{id}_{\mathbb{S}^1}$, which lifts to a generator of $\pi_1(\operatorname{Aut}(\mathcal{P}_k))$. Thus the connecting homomorphism δ_1 vanishes. The argument from Remark 3.4.13 shows precisely that $\pi_0(\operatorname{Diff}(\mathbb{S}^1)_{\mathcal{P}_k})\cong\mathbb{Z}_2$ if and only if $k^2\in K_0$ and that $\pi_0(q)$ is surjective. We thus end up with an exact sequence

$$\operatorname{Fix}_{\pi_0(K)}([k]) \to \pi_0(\operatorname{Aut}(\mathcal{P}_k)) \twoheadrightarrow \begin{cases} \mathbb{Z}_2 & \text{if } k^2\in K_0 \\ \mathbb{1} & \text{else.} \end{cases}$$

Since (5.24) implies that $\operatorname{Diff}(\mathbb{S}^1)_0$ is a $K(1,\mathbb{Z})$, we also have for $n\geq 2$ $\pi_n(\operatorname{Aut}(\mathcal{P}_k))\cong\pi_n(C^\infty_k(\mathbb{S}^1,K))$. $\blacksquare$

Remark 5.4.11. The description of $\mathrm{Aut}(C^\infty_k(\mathbb{S}^1,\mathfrak{k}))$ in Example 5.4.10 should arise out of a general principle, describing the automorphism group for gauge algebras of (flat) bundles, i.e., of bundles of the form

$$\mathcal{P}_\varphi = \widetilde{M} \times K/\sim \quad \text{where} \ \ (m,k) \sim (m\cdot d, \varphi^{-1}(d)\cdot k).$$

Here $\varphi : \pi_1(M) \to K$ is a homomorphism and $\widetilde{M}$ is the simply connected cover of M, on which $\pi_1(M)$ acts canonically (cf. Remark B.3.15). Then

$$\mathfrak{gau}(\mathcal{P}) \cong C^\infty_\varphi(M,\mathfrak{k}) := \{\eta \in C^\infty(\widetilde{M},\mathfrak{k}) : \eta(m\cdot d) = \mathrm{Ad}(\varphi(d))^{-1}.\eta(m)\}.$$

and this description should allow to reconstruct gauge transformations and diffeomorphisms out of the ideals of $C^\infty_\varphi(M,\mathfrak{k})$ (cf. [Le80]). ∎

Problem 5.4.12. Let $\mathcal{P}_\varphi$ be a (flat) principal K-bundle over the closed compact manifold M. Determine the automorphism group $\mathrm{Aut}(\mathfrak{gau}(\mathcal{P}))$. In which cases does it coincide with $\mathrm{Aut}(\mathcal{P})$ (the main point here is the surjectivity of the canonical map $\mathrm{Aut}(\mathcal{P}) \to \mathrm{Aut}(\mathfrak{gau}(\mathcal{P}))$). ∎

The central extension of $\mathfrak{gau}(\mathcal{P})$ from Remark 5.1.5 corresponds to the cocycle $(\eta,\mu) \mapsto [k(\eta, d\mu)]$ on $C^\infty(M,\mathfrak{k})$ from [MN03] in the case of trivial bundles. An interesting generalisation of the cocycle for $\mathfrak{gau}(\mathcal{P})$, that one does not see in the case of mapping algebras (or trivial bundles) is the following.

We first give an example of a finite-dimensional Lie group, for which the universal form $\kappa : \mathfrak{k}\times\mathfrak{k} \to V(\mathfrak{k})$ is *not* K-invariant.

Example 5.4.13 (Non K-invariant universal form). Take $\mathfrak{k} = \mathfrak{sl}_2(\mathbb{C})$ as a real simple Lie algebra. Then complex conjugation induces an automorphism of $\mathfrak{sl}_2(\mathbb{C})$, which leaves invariant the real part of the Cartan–Killing form κ_{CK} and changes the sign of the imaginary part of κ_{CK}. Since κ_{CK} is the universal form of $\mathfrak{sl}_2(\mathbb{C})$, this shows that in general the universal form is not invariant under all automorphisms. More precisely, the universal form κ_{CK} is not invariant under the adjoint action of $K := \mathrm{SL}_2(\mathbb{C}) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $\mathrm{SL}_2(\mathbb{C})$ by complex conjugation. It is *equivariant* with respect to the action of K on $V(\mathfrak{k}) \cong \mathbb{C}$, induced by the $\mathbb{Z}_2$ action on $\mathbb{C}$ by complex conjugation. ∎

The previous example motivates the following generalisation of the cocycle $(\eta,\mu) \mapsto [\kappa_*(\eta, d^A\mu)]$ from Lemma 5.1.3.

Remark 5.4.14. Let $\mathcal{P}$ be a finite-dimensional principal K-bundle over some closed manifold M, Y be a Frećhet space, $\lambda : K \times Y \to Y$ be a smooth action and $\lambda(\mathcal{P})$ be the associated vector bundle. If $\kappa : \mathfrak{k}\times\mathfrak{k} \to Y$ is continuous, bilinear, symmetric and K-*equivariant* and A is a connection 1-from on $\mathcal{P}$, then we set $\mathfrak{z}_M(A,Y) := \Omega^1(M,\lambda(\mathcal{P}))/\, d^A\,\Omega^0(M,\lambda(\mathcal{P}))$ and

$$\widetilde{\omega}_{\kappa,A} : \mathfrak{gau}(\mathcal{P}) \times \mathfrak{gau}(\mathcal{P}) \to \mathfrak{z}_M(A,Y), \quad (\eta,\mu) \mapsto [\kappa_*(\eta, d^A\,\mu)]$$

is cocycle with values in the bundle-valued 1-forms on $\lambda(\mathcal{P})$ (modulo exact 1-forms). That this defines in fact a cocycle is shown exactly as in the case where κ is K-*invariant* in Lemma 5.1.3, where the cocycle has values in Y-valued 1-forms on the base (modulo exact 1-forms). In order to make the target space $\mathfrak{z}_M(A,Y)$ accessible, we have to identify it with some de Rham cohomology space as in Section 5.2. The problem occurring now is that

$$\Omega^0(M,\lambda(\mathcal{P})) \xrightarrow{d^A} \Omega^1(M,\lambda(\mathcal{P})) \xrightarrow{d^A} \ldots$$

is no differential complex since the curvature $(d^A)^2$ of A vanishes only if $\lambda(\mathcal{P})$ is a flat vector bundle. One way around this is to consider cocycles taking values in the twisted cohomology of some flat vector bundle.

In particular, if we take $Y=V(\mathfrak{k})$ and $\kappa:\mathfrak{k}\times\mathfrak{k}\to V(\mathfrak{k})$ to be universal, then K acts on $V(\mathfrak{k})$ in the following way. Since κ is universal, for each $\varphi\in\operatorname{Aut}(\mathfrak{k})$ there exists a linear isomorphism $V(\varphi):V(\mathfrak{k})\to V(\mathfrak{k})$ such that $\kappa\circ(\varphi\times\varphi)=V(\varphi)\circ\kappa$. Since $V(\varphi)$ is unique we have $V(\varphi\circ\psi)=V(\varphi)\circ V(\psi)$ for $\varphi,\psi\in\operatorname{Aut}(\mathfrak{k})$. Thus K acts on $V(\mathfrak{k})$ by

$$V(\mathrm{Ad}):K\times V(\mathfrak{k})\to V(\mathfrak{k}),\quad (k,v)\mapsto V(\mathrm{Ad}(k)).v.$$

and κ is K-equivariant by the construction of the action of K on $V(\mathfrak{k})$. Furthermore, K_0 acts trivially on $V(\mathfrak{k})$, because κ is $\mathfrak{k}$-invariant. In fact, for $x\in\mathfrak{k}$ we have

$$\left.\frac{d}{dt}\right|_{t=0}\kappa(\mathrm{Ad}(\exp(tx)).v,\mathrm{Ad}(\exp(tx).w))=\kappa(\mathrm{ad}(x,v),w)+\kappa(v,\mathrm{ad}(x,w))=0,$$

because κ is $\mathfrak{k}$-invariant, and thus $\kappa(\mathrm{Ad}(\exp(x)).v,\mathrm{Ad}(\exp(x)).w)=\kappa(v,w)$ by the uniqueness of solutions of ordinary differential equations. Since K_0 is generated by $\exp(\mathfrak{k})$, this implies $\kappa(\mathrm{Ad}(k).v,\mathrm{Ad}(k).w)=\kappa(v,w)$ if $k\in K_0$ and thus $\kappa\circ(\mathrm{Ad}(k)\times\mathrm{Ad}(k))=\kappa$. Then the uniqueness of $V(\mathrm{Ad}(k))$ implies $V(\mathrm{Ad}(k))=V(\mathrm{id}_{\mathfrak{k}})=\mathrm{id}_{V(\mathfrak{k})}$ if $k\in K_0$, and hence we get an action

$$\lambda_0:\pi_0(K)\times V(\mathfrak{k})\to V(\mathfrak{k}),\quad [k].v=\mathrm{Ad}(k).v.$$

In addition $\mathcal{P}$ induces a $\pi_0(K)$-bundle $\mathcal{P}_0$ over M, by composing the transition functions $k_{ij}:U_i\cap U_j\to K$ of a cocycle describing $\mathcal{P}$ with the quotient homomorphism $q:K\to K/K_0=\pi_0(K)$ to obtain a cocycle describing the principal $\pi_0(K)$-bundle $\mathcal{P}_0$ over M (cf. Remark B.1.7). This principal bundle is a covering, since the structure group $\pi_0(K)$ is discrete and thus it is in particular flat.

Now the $\pi_0(K)$-action λ_0 induces an associated vector bundle $\lambda_0(\mathcal{P})$. Since this bundle is flat we have a natural covariant derivative and thus a differential complex

$$\ldots\to\Omega^n(M,\lambda_0(\mathcal{P}))\xrightarrow{d}\Omega^{n+1}(M,\lambda_0(\mathcal{P}))\to\ldots.$$

We call the resulting cohomology spaces $H^n(M, \lambda_0(\mathcal{P}))$ the $\lambda_0(\mathcal{P})$-*valued twisted cohomology* of M (cf. [BT82, §1.7]).

By Definition B.3.10, we get a map

$$\kappa_* : \Omega^0(M, \mathrm{Ad}(\mathcal{P})) \times \Omega^1(M, \mathrm{Ad}(\mathcal{P})) \to \Omega^1(M, \lambda_0(\mathcal{P}))$$

and thus

$$\widetilde{\omega}_{\kappa,A} : \mathfrak{gau}(\mathcal{P}) \times \mathfrak{gau}(\mathcal{P}) \to \Omega^1(M, \lambda_0(\mathcal{P}))/\Omega^0(M, \lambda_0(\mathcal{P})), \quad (\eta, \mu) \mapsto [\kappa_*(\eta, d^A \mu)].$$

Now the whole procedure of Section 5.2 can start over again by substituting the ordinary de Rham cohomology $H^1(M, V(\mathfrak{k}))$ with the twisted de Rham cohomology $H^1(M, \lambda_0(\mathcal{P}))$, which is accessible in terms of the group cohomology $H^1(\pi_1(M), V(\mathfrak{k}))$. This leads to further sources of central extensions of gauge groups, which one does not see for trivial bundles. ■

Problem 5.4.15. When does the central extension of $\mathfrak{gau}(\mathcal{P})$, given by the cocycle $\widetilde{\omega}_{\kappa,A}$ from Remark 5.4.14, integrate to a central extension of $\mathrm{Gau}(\mathcal{P})_0$ and how does the corresponding period group look like. Furthermore, if K is not connected, is the central extension of $\mathfrak{gau}(\mathcal{P})$ equivalent to the central extension given in Remark 5.1.5? ■

Appendix A

Notions Of infinite-dimensional Lie theory

A.1 Differential calculus in locally convex spaces

In this section we provide the elementary notions of differential calculus on locally convex spaces and the corresponding notions of infinite-dimensional Lie theory.

We use the same notion for differentiability on open sets and locally convex manifolds as introduced in Section 2.1.

Remark A.1.1 (Some history of differential calculus). The notion of differential calculus that we use dates back to the work of Aristotle Demetrius Michal and Andrée Bastiani in [Mi38], [Mi40] and [Ba64] and is called the Michal–Bastiani Calculus. According to [Ke74], where smooth maps in the Michal-Bastiani sense are called C_c^∞-maps, this notion is the most natural one on locally convex spaces, because it does not involve any assumptions on convergence structures on spaces of linear mappings. Basic results on this calculus can be found in [Mi80] and in [Ha82]. Its first application to infinite-dimensional infinite-dimensional Lie theory has been done by John Willard Milnor in [Mi84], along with many general results and examples. This area is still intensively studied, cf. [Ne06b], [GN07a] and [GN07b]. It has also been extended to arbitrary non-discrete base-fields in [BGN04] and [Gl04]. ■

Remark A.1.2 (Convenient Calculus). We briefly recall the basic definitions underlying the convenient calculus from [KM97]. Let E and F be locally convex spaces. A curve $f : \mathbb{R} \to E$ is called smooth if it is smooth in the sense of Definition 2.1.2. Then the c^∞-topology on E is the final topology induced from all smooth curves $f \in C^\infty(\mathbb{R}, E)$. If E is a Fréchet space, then the c^∞-topology is again a locally convex vector topology which coincides with the original topology [KM97, Theorem 4.11]. If $U \subseteq E$ is c^∞-open then $f : U \to F$ is said to be C^∞ or smooth if

$$f_* \left(C^\infty(\mathbb{R}, U) \right) \subseteq C^\infty(\mathbb{R}, F),$$

e.g., if f maps smooth curves to smooth curves. Remark 2.1.6 implies that each smooth map in the sense of Definition 2.1.2 is smooth in the convenient sense. On the other hand [KM97, Theorem 12.8] implies that on a Fréchet space a smooth map in the convenient sense is smooth in the sense of Definition 2.1.2. Hence for Fréchet spaces the two notions coincide. ■

Definition A.1.3 (Locally convex Lie group). A *locally convex Lie group* (or shortly a *Lie group*) is a group G which is a locally convex manifold such that the multiplication map $m_G : G \times G \to G$ an the inversion map $\iota_G : G \to G$ is smooth. A morphism of locally convex Lie groups is a smooth group homomorphism. ■

Definition A.1.4 (Centred Chart, Convex Subset). Let G be a Lie group modelled on a locally convex topological vector space Y. A chart $\varphi : W \to \varphi(W) \subseteq Y$ with $e \in W$ and $\varphi(e) = 0$ is called a *centred chart.* A subset L of W is called *φ-convex* if $\varphi(L)$ is a convex subset of Y. If W itself is φ-convex, we speak of a *convex centred chart.* ■

Remark A.1.5 (Existence of centred charts). It is clear that every open unit neighbourhood in G contains a φ-convex open neighbourhood for each centred chart φ, because we can pull back any convex open neighbourhood that is small enough from the underlying locally convex vector space along φ to a φ-convex unit neighbourhood.

Typical centred charts arise from the (inverse of the) exponential function for a locally exponential Lie group G (cf. Definition A.1.10). ■

Proposition A.1.6 (Local description of Lie groups). *Let G be a group with a locally convex manifold structure on some subset $U \subseteq G$ with $e \in U$. Furthermore, assume that there exists $V \subseteq U$ open such that $e \in V$, $VV \subseteq U$, $V = V^{-1}$ and*

i) $V \times V \to U$, $(g, h) \mapsto gh$ is smooth,

ii) $V \to V$, $g \mapsto g^{-1}$ is smooth,

iii) for all $g \in G$, there exists an open unit neighbourhood $W \subseteq U$ such that $g^{-1}Wg \subseteq U$ and the map $W \to U$, $h \mapsto g^{-1}hg$ is smooth.

Then there exists a unique locally convex manifold structure on G which turns G into a Lie group, such that V is an open submanifold of G.

Proof. The proof of [Bo89b, Proposition III.1.9.18] carries over without changes.■

Definition A.1.7 (Locally convex Lie algebra). A *locally convex Lie algebra* is a locally convex vector space $\mathfrak{g}$ together with a continuous bilinear alternating map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi Identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

for all $x, y, z \in \mathfrak{g}$. ■

Lemma A.1.8 (Tangent bundle of a Lie group is trivial). *If G is a locally convex Lie group, then the tangent bundle TG of G is trivial, i.e., there is an isomorphism of locally convex manifolds $\Theta : TG \to G \times T_eG$ such that $\Theta|_{T_gG} : T_gG \to \{g\} \times T_eG$ is a linear isomorphism for each $g \in G$.*

Proof. Clearly, $\Theta : TG \to G \times T_eG,\ X_g \mapsto (g, T\lambda_g^{-1}.X_g)$ defines such a global trivialisation. ∎

Remark A.1.9 (The Lie algebra of a locally convex Lie group). A vector field X on a locally convex Lie group G is called *left invariant* if

$$X \circ \lambda_g = T\lambda_g \circ X$$

as mappings $G \to TG$, where $\lambda_g := m_G(g, \cdot) : G \to G$. Clearly, $X \mapsto X(e)$ is an isomorphism between the vector space $\mathcal{V}(G)^l$ of left invariant vector fields on G and T_eG. This endows $\mathcal{V}(G)^l$ with a locally convex vector topology. If X and X' are vector fields on G, then there exists a unique vector filed $[X, X'] \in \mathcal{V}(G)$ determined by the condition that

$$[X, X'].f = X.(X'.f) - X'.(X.f)$$

for each open subset $U \subseteq G$ and all $f \in C^\infty(U, R)$ and $U \subseteq M$ open. Moreover, if X and X' are left invariant, then $[X, X']$ is so. We thus have a bilinear alternating map

$$[\cdot, \cdot] : \mathcal{V}(G)^l \times \mathcal{V}(G)^l \to \mathcal{V}^l(G),$$

which induces a bilinear alternating map on $T_e(G)$. Furthermore, this map is continuous and satisfies the Jacobi identity and thus is a continuous Lie bracket on T_eG. It thus turns T_eG into a locally convex Lie algebras, which we denote by $\mathfrak{g}$. ∎

Definition A.1.10 (Exponential function, locally exponential Lie group). Let G be a locally convex Lie group. The group G is said to have an *exponential function* if for each $x \in \mathfrak{g}$ the initial value problem

$$\gamma(0) = e, \quad \gamma'(t) = T\lambda_{\gamma(t)}(e).x$$

has a solution $\gamma_x \in C^\infty(\mathbb{R}, G)$ and the function

$$\exp_G : \mathfrak{g} \to G, \ \ x \mapsto \gamma_x(1)$$

is smooth. Furthermore, if there exists a zero neighbourhood $W \subseteq \mathfrak{g}$ such that $\exp_G|_W$ is a diffeomorphism onto some open unit neighbourhood of G, then G is said to be *locally exponential*. ∎

Remark A.1.11 (Banach–Lie groups are locally exponential). The Fundamental Theorem of Calculus for locally convex spaces (cf. [Gl02a, Theorem 1.5]) yields that a locally convex Lie group G can have at most one exponential function (cf. [Ne06b, Lemma II.3.5]). If G is a Banach-Lie group (i.e., $\mathfrak{g}$ is a Banach space) , then G is locally exponential due to the existence of solutions of differential equations, their smooth dependence on initial values [La99, Chapter IV] and the Inverse Mapping Theorem for Banach spaces [La99, Theorem I.5.2]. In particular, each finite-dimensional Lie group is locally exponential. ■

Lemma A.1.12 (Locally exponential Lie groups and homomorphisms). *If G and G' are locally convex Lie groups with exponential function, then for each morphism $\alpha : G \to G'$ of Lie groups and the induced morphism $d\alpha(e) : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras, the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G' \\ \uparrow{\scriptstyle \exp_G} & & \uparrow{\scriptstyle \exp_{G'}} \\ \mathfrak{g} & \xrightarrow{d\alpha(e)} & \mathfrak{g}' \end{array}$$

commutes.

Proof. For $x \in \mathfrak{g}$ consider the curve

$$\tau : \mathbb{R} \to G, \quad t \mapsto \exp_G(tx).$$

Then $\gamma := \alpha \circ \tau$ is a curve such that $\gamma(0) = e$ and $\gamma(1) = \alpha\big(\exp_G(x)\big)$ with left logarithmic derivate $\delta^l(\gamma) = d\alpha(e).x$. ■

Remark A.1.13 (Infinite-dimensional Lie theory). Since smooth maps are continuous, each locally convex Lie group G is in particular a topological group. This is one of the main advantages of this approach to infinite-dimensional Lie groups, because it permits the combination of geometric properties from G as a manifold, topological properties from G as a topological space and algebraic properties from the Lie algebra $\mathfrak{g}$ of G in order to develop an infinite-dimensional Lie theory for locally convex Lie groups (cf. [Ne06b]).

One very important fact for this theory is the Fundamental Theorem Of Calculus for locally convex spaces [Gl02a, Theorem 1.5], because is implies that a function is (up to a constant) determined by its derivative. ■

Remark A.1.14 (Complex Lie groups and algebras). If X and Y are complex locally convex spaces and $U \subseteq X$ is open, then f is called *holomorphic* if it is C^1 and the map $df(x) : X \to Y$ is complex linear for all $x \in U$ (cf. [Mi84, p.1027]). In this case, f is automatically smooth if Y is sequentially complete [Ne01, Proposition I.10]. From this notion it is clear what the notion of a *complex locally convex Lie group* (or shortly a *complex Lie group*) is, i.e., a locally convex Lie group, which is in particular smooth, that is modelled on a complex locally convex space such that the group operation are holomorphic in local coordinates.■

A.2 Central extensions of locally convex Lie algebras and groups

In this section we recall the concept of central extensions for topological Lie algebras and locally convex Lie groups.

Definition A.2.1 (Central extensions of Lie algebras). If $\mathfrak{g}$ is a locally convex Lie algebra and $\mathfrak{z}$ is a locally convex vector space, then a *central extension* of $\mathfrak{g}$ by $\mathfrak{z}$ is a short exact sequence

$$\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g}$$

that splits linearly, i.e., there exists a continuous linear section $\alpha : \mathfrak{g} \to \widehat{\mathfrak{g}}$. This extension is said to be trivial if α can be chosen to be a morphism of topological Lie algebras. Two central extensions $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ of $\mathfrak{g}$ by $\mathfrak{z}$ are said to be *equivalent central extensions* if there exists an isomorphism of topological Lie algebras $\varphi : \widehat{\mathfrak{g}}_2 \to \widehat{\mathfrak{g}}_2$ such that the diagram

$$\begin{tikzcd}
\mathfrak{z} \arrow[r] \arrow[d, "\mathrm{id}"'] & \widehat{\mathfrak{g}}_1 \arrow[r, "q_1"] \arrow[d, "\varphi"'] & \mathfrak{g}, \arrow[d, "\mathrm{id}_{\mathfrak{g}}"'] \\
\mathfrak{z} \arrow[r] & \widehat{\mathfrak{g}}_1 \arrow[r, "q_1"] & \mathfrak{g}
\end{tikzcd}$$

commutes, where $\varphi_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ is the map induced by φ on the quotients. Note that a central extension is trivial if and only if it is equivalent to the trivial central extension $\widehat{\mathfrak{g}} = \mathfrak{z} \oplus \mathfrak{g}$. ■

Remark A.2.2 (Central extensions of Lie algebras and cocycles). If $\mathfrak{g}$ is a locally convex Lie algebra, $\mathfrak{z}$ is a locally convex space $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ is a central extension, then the linear section α determines a continuous bilinear alternating mapping

$$\omega_{\widehat{\mathfrak{g}}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}, \quad (x, y) \mapsto [\alpha(x), \alpha(y)] - \alpha([x, y]), \tag{A.1}$$

which satisfies the cocycle condition

$$\omega_{\widehat{\mathfrak{g}}}([x, y], z) + \omega_{\widehat{\mathfrak{g}}}([y, z], x) + \omega_{\widehat{\mathfrak{g}}}([z, x], y) = 0. \tag{A.2}$$

On the other hand, for a $\mathfrak{z}$-valued *cocycle* on $\mathfrak{g}$, i.e., a continuous bilinear alternating map $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ satisfying (A.2), we define a continuous Lie bracket on $\mathfrak{z} \oplus \mathfrak{g}$ by

$$[(z, x), (z', x')] = (\omega(x, x'), [x, x']). \tag{A.3}$$

We denote by $\mathfrak{z} \oplus_\omega \mathfrak{g}$ the topological Lie algebra determined by (A.3), which in turn defines a central extension

$$\mathfrak{z} \hookrightarrow \mathfrak{z} \oplus_\omega \mathfrak{g} \xrightarrow{\mathrm{pr}_2} \mathfrak{g},$$

which we will refer to as $\widehat{\mathfrak{g}}_\omega$. If $\omega_{\widehat{\mathfrak{g}}}$ is the cocycle from (A.1), then $\widehat{\mathfrak{g}}_{\omega_{\widehat{\mathfrak{g}}}}$ and $\widehat{\mathfrak{g}}$ are equivalent, because we have the equivalence $(z,x)\mapsto z+\alpha(x)$. Thus each central extension $\widehat{\mathfrak{g}}$ is equivalent to some $\widehat{\mathfrak{g}}_\omega$ for a cocycle ω. Furthermore, (A.3) implies that two central extensions $\widehat{\mathfrak{g}}_\omega$ and $\widehat{\mathfrak{g}}_{\omega'}$ are equivalent if and only if the corresponding cocycles satisfy

$$\omega(x,x')=\omega'(x,x')+\beta([x,x']) \tag{A.4}$$

for some continuous linear map $\beta:\mathfrak{g}\to\mathfrak{z}$. Thus the second *continuous Lie algebra cohomology*

$$H^2_c(\mathfrak{g},\mathfrak{z}):=\{\omega:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{z}:\omega \text{ is a cocycle }\}/\sim$$

with $\omega\sim\omega'$ if there exists some continuous linear map $\beta:\mathfrak{g}\to\mathfrak{z}$ satisfying (A.4), parametrises the equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{z}$. ∎

Lemma A.2.3 (Automorphisms of central extensions of Lie algebras). *Let $\mathfrak{g}$ be a topological Lie algebra, $\mathfrak{z}$ be a locally convex space and $\omega\in Z^2_c(\mathfrak{g},\mathfrak{z})$. If $\gamma_{\mathfrak{g}}\in\mathrm{Aut}(\mathfrak{g})$, $\gamma_{\mathfrak{z}}\in\mathrm{Lin}(\mathfrak{z})$ and $\alpha\in\mathrm{Lin}(\mathfrak{g},\mathfrak{z})$, then*

$$\varphi:\mathfrak{z}\oplus\mathfrak{g}\to\mathfrak{z}\oplus\mathfrak{g},\quad (z,x)\mapsto(\gamma_{\mathfrak{z}}(z)+\alpha(x),\gamma_{\mathfrak{g}}(x))$$

defines an element of $\mathrm{Aut}(\mathfrak{z}\oplus_\omega\mathfrak{g})$ *if and only if*

$$\omega(\gamma_{\mathfrak{g}}(x),\gamma_{\mathfrak{g}}(x'))=\gamma_{\mathfrak{z}}(\omega(x,x'))+\alpha([x,x'])$$

holds for all $x,x'\in\mathfrak{g}$.

Proof. Unwinding the definitions we get

$$\begin{aligned}(\gamma_{\mathfrak{z}}(\omega(x,x'))+\alpha([x,x']),\gamma_{\mathfrak{g}}([x,x']))&=\varphi((\omega(x,x'),[x,x']))=\varphi([(z,x),(z',x')])\\ &\overset{!}{=}[\varphi(z,x),\varphi(z',x')]=[(\gamma_{\mathfrak{z}}(z)+\alpha(x),\gamma_{\mathfrak{g}}(x)),(\gamma_{\mathfrak{z}}(z')+\alpha(x'),\gamma_{\mathfrak{g}}(x'))]\\ &=(\omega(\gamma_{\mathfrak{g}}(x),\gamma_{\mathfrak{g}}(x')),[\gamma_{\mathfrak{g}}(x),\gamma_{\mathfrak{g}}(x')])=(\omega(\gamma_{\mathfrak{g}}(x),\gamma_{\mathfrak{g}}(x')),\gamma_{\mathfrak{g}}([x,x']))\end{aligned}$$

and the assertion is immediate. ∎

Definition A.2.4 (Central extensions of locally convex Lie groups). Let $\mathfrak{z}$ be a locally convex space, $\Gamma\subseteq\mathfrak{z}$ be a discrete subgroup and G be a connected locally convex Lie group. A *central extension* of G by Z is a short exact sequence

$$Z\hookrightarrow\widehat{G}\xrightarrow{q}\!\!\!\!\rightarrow G \tag{A.5}$$

such that q has local smooth sections (i.e., (A.5) defines a principal Z-bundle over G). This extension is said to be *trivial* if there exists a global smooth section of q that is a morphism Lie groups. Two central extensions $\widehat{G}_1$ and $\widehat{G}_2$ of G by Z

are said to be *equivalent* if there exists an isomorphism of Lie groups $\varphi : \widehat{G}_2 \to \widehat{G}_2$ such that the diagram

$$\begin{array}{ccccc} Z & \longrightarrow & \widehat{G}_1 & \xrightarrow{q_1} & G, \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \varphi}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ Z & \longrightarrow & \widehat{G}_1 & \xrightarrow{q_1} & G \end{array}$$

commutes. Note that a central extension is trivial if and only if it is equivalent to the trivial central extension $\widehat{G} = Z \times G$. ■

Remark A.2.5 (Central extensions of Lie groups and cocycles). If $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ is a central extension, then there exists a section $S : G \to \widehat{G}$ with $S(e_G) = e_{\widehat{G}}$ which is smooth on a unit neighbourhood (take a local sooth section and extend it to a global, not necessarily continuous section). Then S defines a mapping

$$f_{\widehat{G}} : G \times G \to Z, \quad (g, g') \mapsto S(g) \cdot S(g) \cdot S(g \cdot g')^{-1},$$

which is smooth on a unit neighbourhood (because S is so) and satisfies

$$f_{\widehat{G}}(g, e) = f_{\widehat{G}}(e, g) = e \quad \text{and} \quad f_{\widehat{G}}(g, g') + f_{\widehat{G}}(g \cdot g', g'') = f_{\widehat{G}}(g, g' \cdot g'') + f_{\widehat{G}}(g', g''). \tag{A.6}$$

On the other hand, for a Z-valued *cocycle* on G, i.e., a map $f : G \times G \to Z$ that is smooth on a unit neighbourhood and satisfies (A.6), we define a multiplication

$$(Z \times G) \times (Z \times G) \to (Z \times G), \quad ((z, g), (z', g')) \mapsto (z + z' + f(g, g'), g \cdot g'), \tag{A.7}$$

which turns $Z \times G$ into a locally convex Lie group. In fact, (A.7) defines a group multiplication because of (A.6) and then Proposition A.1.6 provides a locally convex Lie group structure on $Z \times G$, where condition *iii*) there is satisfied, because G is assumed to be connected. We then denote by $Z \times_f G$ the locally convex Lie group determined by (A.7), which in turn defines a central extension

$$Z \hookrightarrow Z \times_f G \xrightarrow{\mathrm{pr}_2} G,$$

which we call $\widehat{G}_f$. If $f_{\widehat{G}}$ is the cocycle from (A.6), then $Z \times_{f_{\widehat{G}}}$ and $\widehat{G}$ are equivalent, because we have the equivalence $(z, g) \mapsto (z + S(g), g)$. Thus, each central extension $\widehat{G}$ is equivalent to some $\widehat{G}_f$ for a cocycle f. Furthermore, (A.7) implies that two central extensions $\widehat{G}_f$ and $\widehat{G}_{f'}$ are equivalent if and only if the corresponding cocycles satisfy

$$f(g, g') - f'(g, g') = h(g) + h(g') - h(g \cdot g') \tag{A.8}$$

for a map $h : G \to Z$ that is smooth on a unit neighbourhood in G. Thus the second *smooth Lie group cohomology*

$$H^2_s(G, Z) := \{f : G \times G \to Z : f \text{ is a cocycle}\}/\sim$$

with $f \sim f'$ if there exists a map $h : G \to Z$ satisfying (A.8), parametrises the equivalence classes of central extensions of G by Z. ■

Remark A.2.6 (From Lie group extensions to Lie algebra extensions). Let $\mathfrak{z}$ be a locally convex space,$\Gamma \subseteq \mathfrak{z}$ be a discrete subgroup and G be a connected locally convex Lie group. Furthermore, let

$$Z := \mathfrak{z}/\Gamma \hookrightarrow Z \times_f G \twoheadrightarrow G$$

be a central extension, which is given by a cocycle $f : G \times G \to Z$ that is smooth on a unit neighbourhood and let $\widehat{\mathfrak{g}}$ be the Lie algebra of $Z \times_f G$. Because the quotient map $\mathfrak{z} \to Z$ has smooth local sections we can lift f to a map $f_{\mathfrak{z}} : G \times G \to \mathfrak{z}$ that is still smooth on a unit neighbourhood. We thus have

$$Df : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}, \quad (x, x') \mapsto d^2 f_{\mathfrak{z}}(x, y) - d^2 f_{\mathfrak{z}}(y, x)$$

if we identify $\mathfrak{g}$ with $T_e G$. Furthermore, Df is a Lie algebra cocycle and we have that $\widehat{\mathfrak{g}}$ is equivalent to $\mathfrak{z} \oplus_{Df} \mathfrak{g}$ as central extension [Ne02a, Lemma 4.6]. Since equivalent Lie group extensions lead to equivalent Lie algebra extensions we thus have a well-defined map

$$D : H^2_s(G, Z) \to H^2_c(\mathfrak{g}, \mathfrak{z}), \quad [f] \mapsto [Df].$$

If $[\omega]$ in $H^2_c(\mathfrak{g}, \mathfrak{z})$ is in the image of D, i.e., if there exists a central extension of Lie groups such that the corresponding central extension of Lie algebras is equivalent to $\widehat{\mathfrak{g}}_\omega$, then we say that the central extension $\widehat{\mathfrak{g}}_\omega$ *integrates* to a central extension of Lie groups. ■

A.3 Actions of locally convex Lie groups

In this section we provide the elementary notions of actions of infinite-dimensional Lie groups on locally convex manifolds.

Definition A.3.1 (Smooth actions of Lie groups). If G is a locally convex Lie group and M is a locally convex manifold, then a smooth map $\lambda : G \times M \to M$, $(g, s) \mapsto g.s$ is called a *smooth action* of G on M if the map $\lambda_g : M \to M$, $s \mapsto g.s$ is a diffeomorphism for each $g \in G$ and $G \ni g \mapsto \lambda_g \in \mathrm{Diff}(M)$ is a homomorphism of abstract groups.

If, moreover, $M = H$ is a locally convex Lie group and each λ_g is an element of $\mathrm{Aut}(H)$, then we call the action a *smooth automorphic action*. Furthermore, if $H = Y$ is a locally convex space and each λ_g is an element of $\mathrm{GL}(Y)$, then we call the action a *smooth linear action*. Finally, if $Y = \mathfrak{k}$ is a locally convex Lie algebra and each λ_g is an element of $\mathrm{Aut}(\mathfrak{k})$, then we call the action also a *smooth automorphic action.* ■

Remark A.3.2 (Adjoint action). A locally convex Lie group G acts in a natural way on its Lie algebra $\mathfrak{g}$ by

$$\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{Ad}(g).x = Tc_g(x),$$

where $c_g : G \to G$ denotes the conjugation map $h \mapsto g \cdot h \cdot g^{-1}$ and $\mathfrak{g}$ is identified with T_eG. This action is in particular smooth and automorphic. ■

Lemma A.3.3 (Actions need only be smooth on unit neighbourhoods). *Let G be a locally convex Lie group, M be a locally convex manifold and $\lambda : G \times M \to M$ be an abstract action, i.e., $\lambda_g \in \mathrm{Diff}(M)$ for all $g \in G$ and $G \ni g \mapsto \lambda_g \in \mathrm{Diff}(M)$ is a homomorphism of abstract groups. Then λ is smooth if and only if there exists an open unit neighbourhood $U \subseteq G$ such that $\lambda|_{U \times M}$ is smooth.*

Proof. For each $g \in G$, let U_g be an open neighbourhood of g such that $g^{-1} \cdot x \in U$ for all $x \in U_g$. Then $\lambda(x, v) = \lambda(g, \lambda(g^{-1} \cdot x, v))$ implies that $\lambda|_{U_g \times M} = \lambda_g \circ \lambda \circ (\lambda_{g^{-1}} \times \mathrm{id}_M)$ is smooth. ■

Lemma A.3.4 (Smoothness criterion for automorphic actions). *Let G and H be locally convex Lie groups, and $\lambda : G \times H \to H$ be an automorphic action of abstract groups., i.e., $\lambda_g \in \mathrm{Aut}(H)$ for all $g \in G$ and $G \ni g \mapsto \lambda_g \in \mathrm{Aut}(H)$ is a homomorphism of abstract groups. Then λ is smooth if and only if the orbit maps*

$$G \ni g \mapsto \lambda(g, h) \in H$$

are smooth for each $h \in H$ and there exists an open unit neighbourhood $U \subseteq H$ such that $\lambda|_{G \times U}$ is smooth.

Proof. For each $h \in H$, let U_h be an open neighbourhood of g such that $h^{-1} \cdot x \in U$ for all $x \in U_h$. Then $\lambda(g, h') = \lambda(g, h) \cdot \lambda(g, h^{-1} \cdot h'))$ implies that $\lambda|_{G \times U_h}$ is smooth, because $\lambda(g, h)$ depends smoothly on g for fixed h. ■

Proposition A.3.5 (Automorphic actions on Lie algebra extension). *Let H be a locally convex Lie group and $\widehat{\mathfrak{g}}_\omega$ be a central extension, given by some cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ as in Remark A.2.2. If $H \times \mathfrak{z} \to \mathfrak{z}$, $(h, z) \mapsto h.z$ is a linear action, $H \times \mathfrak{g} \to \mathfrak{g}$, $(h, x) \mapsto h.x$ is an automorphic action and $R : H \times \mathfrak{g} \to \mathfrak{z}$ is a map, then*

$$H \times \mathfrak{z} \times \mathfrak{g} \to \mathfrak{z} \times \mathfrak{g}, \quad h.(z, x) = (h.z + R(h, x), h.x) \tag{A.9}$$

defines a smooth automorphic action of H on $\mathfrak{z} \oplus_\omega \mathfrak{g}$ if and only if

$$h.R(h', x) + R(h, h'.x) = R(h \cdot h', x), \tag{A.10}$$

for all $h, h' \in H$ and $x \in \mathfrak{g}$, the restriction of R to $U \times (\mathfrak{z} \oplus \mathfrak{g})$ is smooth for some open unit neighbourhood $U \subseteq H$ and

$$\omega(h.x, h.x') = h.\omega(x, x') + R(h, [x, x']) \tag{A.11}$$

for all $h \in H$ and $x, x' \in \mathfrak{g}$.

Proof. A direct computation yields that (A.10) is equivalent to the condition that (A.9) defines an abstract action. Then the smoothness of the action follows from Lemma A.3.3 and the assertion follows from Lemma A.2.3. ■

Remark A.3.6 (Cocycle for group actions). Let G and H be locally convex Lie groups and $\lambda : G \times H \to H$ be a smooth action. A *crossed homomorphism* or 1-*cocycle* is a smooth map $f : G \to N$ with

$$f(g \cdot h) = f(g) \cdot g.f(h) \text{ for all } g, h \in H,$$

which is equivalent to $(f, \mathrm{id}_G) : G \to H \rtimes G$ being a group homomorphism. We note that in view of Lemma A.3.3 this implies, in particular, that for a 1-cocycle, smoothness on an identity neighbourhood is equivalent to global smoothness. ■

Definition A.3.7 (Derived action). If G is a locally convex Lie group, Y is a locally convex space and $\lambda : G \times Y \to Y$ is a smooth action, then

$$\dot{\lambda} : \mathfrak{g} \times Y \to Y, \quad x.y = d\lambda(e, y)(x, 0)$$

is called the *derived action*. In the special case of the adjoint action of G on $\mathfrak{g}$, we get $\dot{\lambda}(x, y) = \mathrm{ad}(x, y) = [x, y]$. ■

Definition A.3.8 (Left logarithmic derivative). If M is a locally convex manifold with corners, G is a locally convex Lie group and $f \in C^\infty(M, G)$, then the *left logarithmic derivative* $\delta^l(f) \in \Omega^1(M, \mathfrak{g})$ of f is defined to be

$$\delta^l(f).X_m := T\lambda_{f^{-1}(m)}(Tf(X_m)).$$

Is is simply the pull-back $f^*\kappa_G$ of the Maurer–Cartan form $\kappa_G : TG \to T_eG$, $X_g \mapsto T\lambda_{g^{-1}}(X_g)$ to M along f. ■

Lemma A.3.9 (Product rule for left logarithmic derivative). *If M is a locally convex manifold with corners, G is a locally convex Lie group and $f, g \in C^\infty(M, G)$, then*

$$\delta^l(f \cdot g) = \delta^l(g) + \mathrm{Ad}(g)^{-1}.\delta^l(f)$$

and in particular $\delta^l(f^{-1}) = -\mathrm{Ad}(f).\delta^l(f)$.

Proof. This follows from the definition and an elementary calculation. ■

Lemma A.3.10 (Product rule for pointwise action). *Let M be a smooth locally convex manifold with corners, G be a locally convex Lie group and $\lambda : G \times Y \to Y$ be a smooth linear action on the locally convex space Y. If $h : M \to G$ and $f : M \to Y$ are smooth, then we have*

$$d\left(\lambda(h).f\right).X_m = \lambda(h).(df.X_m) + \dot{\lambda}\left(\mathrm{Ad}(h).\delta^l(h).X_m\right).\left(\lambda(h(m)).f(m)\right) \quad \text{(A.12)}$$

with $\lambda(h^{-1}).f : M \to E$, $m \mapsto \lambda\left(h(m)^{-1}\right).f(m)$. If $\lambda = \mathrm{Ad}$ is the adjoint action of G on $\mathfrak{g}$, then we have

$$d\left(\mathrm{Ad}(h).f\right).X_m = \mathrm{Ad}(h).(df.X_m) + \mathrm{Ad}(h).\left[\delta^l(h).X_m, f(m)\right]$$

Proof. We write $\lambda(h,f)$ instead of $\lambda(h).f$, interpret it as a function of two variables, suppress the dependence on m and calculate

$$\begin{aligned} d\left(\lambda(h,f)\right)(X_m,X_m) &= d\left(\lambda(h,f)\right)\big((0_m,X_m)+(X_m,0_m)\big) \\ &= d_2\left(\lambda(h,f)\right)(X_m) + d_1\left(\lambda(h).f\right)(X_m) \\ &= \lambda(h,df(X_m)) + d\lambda(\cdot,f).Th(X_m) \\ &= \lambda(h).(df(X_m)) + d\lambda(\cdot,f).T(\lambda_h\circ\lambda_{h^{-1}}\circ\lambda_h\circ\lambda_{h^{-1}}\circ h)(X_m) \\ &= \lambda(h).(df(X_m) + d\left(\lambda(\cdot,f)\circ\lambda_G(h)\right).\mathrm{Ad}(h).\delta^l(h)(X_m) \\ &= \lambda(h).(df(X_m)) + \dot{\lambda}\left(\mathrm{Ad}(h).\delta^l(h)(X_m),\lambda(h(m),f(m))\right), \end{aligned}$$

d_2) denotes the differential of λ with respect to the first (respectively second) variable, keeping constant the second (respectively first) variable. ■

Appendix B

Notions of bundle theory

B.1 Vector- and Principal Bundles

In this section we provide the basic concepts of continuous and smooth vector bundles. In particular, we focus on a description of principal bundles in terms of transition functions (or cocycles), because this is the picture of principal bundles we mostly use.

Throughout the thesis, we always assume that the base spaces of the bundles under consideration are *connected*.

Definition B.1.1 (Continuous vector bundle). Let X be a topological space and Y be a locally convex space. A *continuous vector bundle* over X with fibre Y (or shortly a *continuous vector bundle*) is a topological space E together with continuous map $\xi : E \to X$ such that each fibre $E_x := \xi^{-1}(x)$ is a locally convex space and that for each point in X there exists an open neighbourhood U, called a *trivialising neighbourhood*, and a homeomorphism

$$\Theta : \xi^{-1}(U) \to U \times Y,$$

called *local trivialisation*, such that $\mathrm{pr}_1 \circ \Theta = \xi|_{\xi^{-1}(U)}$ and that $\Theta|_{E_x}$ is an isomorphism of topological vector spaces from E_x to $\{x\} \times Y \cong Y$ for each $x \in U$. We often refer to a vector bundle as a tuple $(Y, \xi : E \to X)$ with the calligraphic letter $\mathcal{E}$. If $\mathcal{E}$ and $\mathcal{E}'$ are two vector bundles, then a *morphism of vector bundles* is a continuous map $f : E \to E'$ such that $f(E_{\xi(e)}) \subseteq E_{\xi(f(e))}$ and $f|_{E_{\xi(e)}}$ is a continuous linear map for each $e \in E$.

A *continuous section* of $\mathcal{E}$ is a continuous map $\sigma : X \to E$, which satisfies $\xi \circ \sigma = \mathrm{id}_X$ and we denote by $S_c(\mathcal{E})$ the space of all continuous sections. If $U \subseteq X$ is a subset, then $\mathcal{E}_U = (Y, \xi|_{\xi^{-1}(U)} : \xi^{-1}(U) \to U)$ denotes the restricted vector bundle over U and $S_c(\mathcal{E}_U)$ is correspondingly the space of sections defined on U. ■

Remark B.1.2 (Transition functions in vector bundles). If $\mathcal{E}$ is a continuous vector bundle with fibre Y and U and U' with $U \cap U' \neq \emptyset$ are two trivialising neighbourhoods, then we have for each $x \in U \cap U'$ an isomorphism

$\varphi_x := \Theta'\big(\Theta^{-1}(x,\cdot)\big) \in \mathrm{GL}(Y)$ induced from the homeomorphism

$$(U\cap U')\times Y \to (U\cap U')\times Y \quad (x,v)\mapsto \Theta'\big(\Theta^{-1}(x,v)\big).$$

Since we have in general no nice topology on $\mathrm{GL}(Y)$ if Y fails to be a Banach space, it does not make sense to put any requirements on the continuity of the map $x\mapsto\varphi_x$.

Furthermore, if K is a topological group acting continuously on Y (i.e., K acts on Y as an abstract group and $K\times Y\to Y$, $(k,y)\mapsto k.y$ is continuous), then $\mathcal{E}$ is a *vector K-bundle* if the local trivialisations can be chosen such that for each pair of trivialising neighbourhoods U and U', there exists a continuous mapping

$$k_{UU'}: U\cap U'\to K$$

with $\varphi_x(y)=k_{UU'}(x).y$ for all $y\in Y$ and $x\in U\cap U'$. ■

Definition B.1.3 (Smooth vector bundle). If E and M are manifolds with corners, then a continuous vector bundle $\xi: E\to M$ with fibre Y is a *smooth vector bundle* if all local trivialisations can be chosen to be diffeomorphisms. If K is a Lie group acting smoothly on Y, then a continuous K-vector bundle ξ is a *smooth vector K-bundle* if the $k_{UU'}$ from Remark B.1.2 can be chosen to be smooth. ■

Definition B.1.4 (Continuous principal bundle). Let K be a topological group. If X is a topological space, then a *continuous principal K-bundle over X* (or shortly a *continuous principal K-bundle*) is a topological space P together with a continuous right action $\rho: P\times K\to P$, $(p,k)\mapsto p\cdot k$ and a map $\pi: P\to X$ such that for each $x\in X$ there exists an open neighbourhood U, called a *trivialising neighbourhood*, such that there exists a homeomorphism

$$\Theta:\pi^{-1}(U)\to U\times K, \tag{B.1}$$

called *local trivialisation*, satisfying $\mathrm{pr}_1\circ\Theta=\pi|_{\pi^{-1}(U_i)}$ and $\Theta(p\cdot k)=\Theta(p)\cdot k$, where K acts on $U\times K$ by right multiplication in the second factor. An arbitrary subset $A\subseteq X$ is called *trivialising* if it has a neighbourhood which is trivialising. We often refer to a continuous principal bundle as a tuple $(K,\pi: P\to X)$ by the calligraphic letter $\mathcal{P}$, where we assume the action of K on the domain of π to be given implicitly. If confusion with homotopy groups could occur, we denote the bundle projection by η instead of π.

A *morphism of continuous principal K-bundles* or a *continuous bundle map* between two continuous principal K-bundles $\mathcal{P}$ and $\mathcal{P}'$ is a continuous map $f: P\to P'$ satisfying $\rho_k'\circ f=f\circ\rho_k$, where ρ_k and ρ_k' are the right actions of $k\in K$ on P and P'. Since the above definition implies in particular $X\cong P/K$ and $X'\cong P'/K$, we obtain an induced map $f_X: X\cong P/K\to X'\cong P'/K$ given by $f_X(p\cdot K):=f(p)\cdot K$. Furthermore, if $X=X'$, then we call f a *bundle equivalence* if it is an isomorphism and $f_X=\mathrm{id}_X$. ■

Remark B.1.5 (Sections define local trivialisations). Let $\mathcal{P} = (K, \pi : P \to M)$ be a continuous principal bundle. If $U \subseteq X$ is open or closed, then a continuous map $\sigma : U \to P$ with $\pi \circ \sigma = \mathrm{id}_U$ is a *continuous section*. In particular, if $U \subseteq X$ is a trivialising neighbourhood, then the corresponding trivialisation $\Theta : \pi^{-1}(U) \to U \times K$ determines a continuous section

$$\sigma_\Theta : U \to P, \quad \sigma(x) = \Theta^{-1}(x, e).$$

Conversely, if $\sigma : U \to P$ is a continuous section of π, then this defines a local trivialisation as follows. For each $p \in \pi^{-1}(U)$ we can write $p = \sigma(\pi(p)) \cdot k_\sigma(p)$ for some $k_\sigma(p) \in K$. This defines a continuous map $k_\sigma : \pi^{-1}(U) \to K$, because $k_\sigma(p) = \mathrm{pr}_2(\Theta(\sigma(p)))^{-1} \cdot \mathrm{pr}_2(\Theta(p))$. We thus have a local trivialisation

$$\Theta_\sigma : \pi^{-1}(U) \to U \times K, \quad p \mapsto (\pi(p), k_\sigma(p)).$$

Since $\Theta_{\sigma_\Theta} = \Theta$ and $\sigma_{\Theta_\sigma} = \sigma$, we have a one-to-one correspondence between local trivialisations and continuous local sections of π. ■

Definition B.1.6 (Trivialising system). Let $\mathcal{P} = (K, \pi : P \to X)$ be a continuous principal K-bundle. If $(U_i)_{i \in I}$ is an open cover of X by trivialising neighbourhoods and $(\sigma_i : U_i \to P)_{i \in I}$ is a collection of continuous sections, then the collection $\mathcal{U} = (U_i, \sigma_i)_{i \in I}$ is called an *continuous open trivialising system* of $\mathcal{P}$.

If $(\overline{U}_i)_{i \in I}$ is a closed cover of X by trivialising sets and $(\sigma_i : \overline{U}_i \to P)_{i \in I}$ is a collection of continuous sections, then the collection $\overline{\mathcal{U}} = (\overline{U}_i, \sigma_i)_{i \in I}$ is called a *continuous closed trivialising system* of $\mathcal{P}$.

If $\mathcal{U} = (U_i, \sigma_i)_{i \in I}$ and $\mathcal{V} = (V_j, \tau_j)_{j \in J}$ are two continuous open trivialising systems of $\mathcal{P}$, then $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if there exists a map $J \ni j \mapsto i(j) \in I$ such that $V_j \subseteq U_{i(j)}$ and $\tau_j = \sigma_{i(j)}\big|_{V_j}$, i.e., $(V_j)_{j \in J}$ is a refinement of $(U_i)_{i \in I}$ and the sections τ_j are obtained from the section σ_i by restrictions.

If $\mathcal{U} = (U_i, \sigma_i)_{i \in I}$ is a continuous open trivialising system and $\overline{\mathcal{V}} = (\overline{V}_j, \tau_j)_{j \in J}$ is a continuous closed trivialising system, then $\overline{\mathcal{V}}$ is a *refinement* of $\mathcal{U}$ if there exists a map $J \ni j \mapsto i(j) \in I$ such that $\overline{V}_j \subseteq U_{i(J)}$ and $\tau_j = \sigma_{i(j)}\big|_{\overline{V}_j}$ and vice versa. ■

Remark B.1.7 (Principal bundles and Cocycles). If $\mathcal{P}$ is a continuous principal K-bundle over X, and U and U' are open trivialising neighbourhoods with $U \cap U' \neq \emptyset$, then the corresponding local trivialisations, given by sections $\sigma_U : U \to P$ and $\sigma_{U'} : U' \to P$, define continuous mappings $k_{UU'} : U \cap U' \to K$ by

$$k_{UU'}(x) = k_{\sigma_U}(\sigma_{U'}(x)) \quad \text{or equivalently} \quad \sigma_U(x) \cdot k_{UU'}(x) = \sigma_{U'}(x), \tag{B.2}$$

called *transition functions*. They satisfy the *cocycle condition*

$$k_{UU}(x) = e \text{ for } x \in U \quad \text{and} \quad k_{UU'}(x) \cdot k_{U'U''}(x) \cdot k_{U''U}(x) = e \text{ for } x \in U \cap U' \cap U'', \tag{B.3}$$

for any third continuous section $\sigma'' : U'' \to P$ with open $U'' \subseteq X$. If $\mathcal{U} = (U_i, \sigma_i)_{i\in I}$ is a continuous open trivialising system, we thus have a collection $\mathcal{K}_{\mathcal{P}} := (k_{ij} : U_i \cap U_j \to K)_{i,j\in I}$ of continuous functions satisfying (B.3).

On the other hand, if $(U_i)_{i\in I}$ is an open cover of X, then each collection $\mathcal{K} = (k_{ij} : U_i \cap U_j \to K)_{i,j\in I}$ of continuous maps satisfying (B.3) is called a *continuous cocycle*. It defines a continuous principal K-bundle $\mathcal{P}_{\mathcal{K}}$ over X if we set

$$P_{\mathcal{K}} = \bigcup_{i\in I} \{i\} \times U_i \times K / \sim \ \text{ with } ((i,x,k) \sim (j,x',k')) :\Leftrightarrow (x = x' \text{ and } k_{ji}(x) \cdot k = k').$$

Then a bundle projection $\pi : P_{\mathcal{K}} \to X$ is given by $[i,x,k] \mapsto x$, $(U_i)_{i\in I}$ is a cover by trivialising open sets with local trivialisations given by

$$\Theta_i : \bigcup_{x\in U_i} [(i,x,k)] \to U_i \times K, \quad [(i,x,k)] \mapsto (x,k)$$

and the K-action is given by $([(i,x,k)],k') \mapsto [(i,x,kk')]$. Thus $(U_i, \tau_i)_{i\in I}$ with $\tau_i(x) = [(i,x,e)]$ is a continuous open trivialising system of $\mathcal{P}_{\mathcal{K}}$. Since $\mathcal{P}_{\mathcal{K}_{\mathcal{P}}}$ is equivalent to $\mathcal{P}$ by the equivalence $[(U,x,k)] \mapsto \sigma_U(x) \cdot k$, each principal K-bundle may equivalently be described by such a collection of continuous functions $\mathcal{K}$. ■

Lemma B.1.8 (Forcing transition functions into open covers). *Let X be a compact space, K be topological group and $(O_\ell)_{\ell\in L}$ be an open cover of K. If $\mathcal{P}$ is a continuous principal K-bundle over X, then for each continuous open trivialising system $\mathcal{U} = (U_i, \sigma_i)_{i=1,\ldots,n}$ there exists a refinement $\mathcal{V} = (V_s, \tau_s)_{s=1,\ldots,r}$ such that for each transition function $k_{st} : V_s \cap V_t \to K$ of $\mathcal{V}$ we have $k_{st}(V_s \cap V_t) \subseteq O_\ell$ for some $\ell \in L$.*

Proof. Let $\kappa_{ij} : U_i \cap U_j \to K$ be the transition functions of $\mathcal{U}$. Furthermore, let $V'_1, \ldots, V'_m$ be an open cover of X such that for each $q \in \{1,\ldots,m\}$ we have $\overline{V'}_q \subseteq U_{i(q)}$ for some $i(q) \in \{1,\ldots,n\}$. By replacing $\mathcal{U}$ by the refinement $(U_{i(q)}, \sigma_{i(q)})_{q=1,\ldots,m}$ we may thus assume $\overline{V'}_i \subseteq U_i$.

For each pair (i,j) with $1 \leq i,j, \leq m$, the open cover $(O_\ell)_{\ell\in L}$ pulls back to an open cover $(\widetilde{O}_\ell^{(i,j)})_{\ell\in L}$ of $U_i \cap U_j$, i.e., $\widetilde{O}_\ell^{(i,j)} := \kappa_{ij}^{-1}(O_\ell)$. Then each $x \in \overline{V'}_i \cap \overline{V'}_j$ has an open neighbourhood $U_x^{(i,j)}$ such that $U_x^{(i,j)} \subseteq V'_q$ for some $q \in \{1,\ldots,m\}$ and $U_x^{(i,j)} \subseteq \widetilde{O}_\ell^{(i,j)}$ for some $\ell \in L$. Then

$$\mathfrak{V}_{(i,j)} := (V'_1 \backslash (\overline{V'}_i \cap \overline{V'}_j), \ldots, V'_m \backslash (\overline{V'}_i \cap \overline{V'}_j), (U_x^{(i,j)})_{x\in \overline{V'}_i \cap \overline{V'}_j})$$

is an open cover of X and each set of this cover is contained in some V'_q.

Now take a common refinement $V_1, \ldots, V_r$ of all the open covers $\mathfrak{V}_{(i,j)}$ for $1 \leq i,j \leq m$. That means, that for each (i,j) and each $s \in \{1,\ldots,r\}$ we have that V_s is contained in one of the open sets of the cover $\mathfrak{V}_{(i,j)}$. Note that this is possible since for each two covers $(Q_s)_{s\in S}$ and $(R_t)_{t\in T}$ we have $(Q_s \cap R_t)_{(s,t)\in S\times T}$ as a common refinement. Since X is compact there exists a finite subcover $V_1, \dots, V_r$ of the common refinement of all $\mathfrak{V}_{(i,j)}$.

Now for each $s \in \{1, \dots, r\}$ we have that V_s is contained in some $V'_{i(s)}$ for some $i(s) \in \{1, \dots, m\}$ and we thus have $V_s \cap V_t \subseteq V'_{i(s)} \cap V'_{i(t)}$. We claim that $V_s \cap V_t$ is contained in one $U_x^{(i(s),i(t))}$ if $V_s \cap V_t \neq \emptyset$. First, recall that V_s is contained in one of the open sets of $\mathfrak{V}_{(i(s),i(t))}$, and the same holds for V_t. The claim is trivially true if V_s or V_t are contained in one $U_x^{(i(s),i(t))}$, so assume $V_s \subseteq V'_q \backslash (\overline{V'}_{i(s)} \cap \overline{V'}_{i(t)})$ and $V_t \subseteq V'_{q'} \backslash (\overline{V'}_{i(s)} \cap \overline{V'}_{i(t)})$ for some $q, q' \in \{1, \dots, m\}$. Then

$$V_s \cap V_t \subseteq (V'_q \cap V'_{q'}) \backslash (\overline{V'}_{i(s)} \cap \overline{V'}_{i(t)}) \text{ and } V_s \cap V_t \subseteq V'_{i(s)} \cap V'_{i(t)} \subseteq \overline{V'}_{i(s)} \cap \overline{V'}_{i(t)}$$

imply $V_s \cap V_t = \emptyset$ and the claim is established.

We now set $\tau_s := \sigma_{i(s)}\big|_{V_s}$ for $s \in \{1, \dots, r\}$. Then $\mathcal{V} := (V_s, \tau_s)_{s=1,\dots,r}$ is a continuous open trivialising system of $\mathcal{P}$, which is a refinement of $\mathcal{U}$. Denote the transition functions of $\mathcal{V}$ by $k_{st} : U_s \cap U_t \to K$. Since the sections of $\mathcal{V}$ are given by restricting the sections of $\mathcal{U}$ and the sections determine the transition functions by $\sigma_s \cdot k_{st} = \sigma_t$, we have $k_{st} = \kappa_{i(s)i(t)}\big|_{V_s \cap V_t}$. We have seen before that if $V_s \cap V_t \neq \emptyset$, then $V_s \cap V_t \subseteq U_x^{(i(s),i(t))}$ for some $x \in \overline{V'}_{i(s)} \cap \overline{V'}_{i(t)}$. Since $U_x^{(i(s),i(t))} \subseteq \widetilde{O}_\ell^{(i(s),i(t))}$ for some $\ell \in L$ we thus have

$$k_{st}(V_s \cap V_t) \subseteq k_{st}(\widetilde{O}_\ell^{(i(s),i(t))}) = \kappa_{i(s)i(t)}(\widetilde{O}_\ell^{(i(s),i(t))}) \subseteq O_\ell. \qquad \blacksquare$$

Remark B.1.9 (Equivalences of principal bundles and cocycles). Let K be a topological group. If X is a topological space and $(U_i)_{i \in I}$ is an open cover of a X, then a collection $\mathcal{K} = (k_{ij} : U_i \cap U_j \to K)_{i,j \in I}$ of continuous maps satisfying (B.3) is called a K-valued cocycle on X. Two such cocycles $\mathcal{K}$ and $\mathcal{K}'$ are said to be equivalent if there exists a common refinement $(V_j)_{i \in J}$ of their open covers together with two functions $f : J \to I$ and $f' : J \to I'$ such that $V_j \subseteq U_{f(j)}$ and $V_j \subseteq U'_{f'(j)}$ for all $j \in J$ and a collection $\mathcal{G} = (g_j : V_j \to K)_{j \in J}$ of continuous functions satisfying

$$g_j^{-1}(x) \cdot k_{f(j)f(j')}(x) \cdot g_{j'}(x) = k'_{f'(j)f'(j')}(x)$$

for all $x \in V_j \cap V_{j'}$. If $\mathcal{P}_{\mathcal{K}}$ and $\mathcal{P}_{\mathcal{K}'}$ are the associated principal K-bundles over X, then this defines a continuous bundle equivalence $g_{\mathcal{G}}$ between $\mathcal{P}_{\mathcal{K}}$ and $\mathcal{P}_{\mathcal{K}'}$ by setting

$$g_{\mathcal{G}} : P_{\mathcal{K}} \to P_{\mathcal{K}'}, \quad [(f(j), x, k)] \mapsto [(f'(j), x, g_j(x) \cdot k)].$$

Conversely, if $\mathcal{P}_{\mathcal{K}}$ and $\mathcal{P}_{\mathcal{K}'}$ are two principal K-bundles over X, given by two cocycles $\mathcal{K}$ and $\mathcal{K}'$, then there exists an open cover $(V_i)_{i \in I}$ which is a common refinement of the open covers $(U_i)_{i \in I}$ and $(U'_{i'})_{i' \in I'}$ underlying $\mathcal{K}$ and $\mathcal{K}'$. In fact,

$$(U_i \cap U'_{i'})_{(i,i') \in I \times I'}$$

is such a cover and, we assign to it the functions $f = \mathrm{pr}_1$ and $f' = \mathrm{pr}_2$. Then a bundle equivalence $g : P_{\mathcal{K}} \to P_{\mathcal{K}'}$ defines for each $(i, i') \in I \times I'$ a continuous map

$$g'_{(i,i')} : U_i \cap U'_{i'} \times K \to K \text{ by } g([(i, x, k)]) = [(i', x, g'_{(i,i')}(x, k))]. \tag{B.4}$$

Sine g is assumed to satisfy $g(p \cdot k) = g(p) \cdot k$, we have $g'_{(i,i')}(x, k) = g'_{(i,i')}(x, e) \cdot k$. If we set $g_{(i,i')}(x) := g'_{(i,i')}(x, e)$, we obtain a collection of continuous maps

$$\mathcal{G}_g := (g_{(i,i')} : U_i \cap U'_{i'} \to K)_{(i,i') \in I \times I'}$$

satisfying

$$k_{j'i'}(x) \cdot g_{(i,i')}(x) = g_{(j,j')}(x) \cdot k_{ji}(x) \text{ for all } x \in U_i \cap U_{i'} \cap U_j \cap U_{j'}, \tag{B.5}$$

because $[(i, x, k)] = [(j, x, k_{ji}(x)k)]$ has to be mapped to the same element of $P_{\mathcal{K}}$ by g. Since $\mathcal{G} = \mathcal{G}_{g_{\mathcal{G}}}$ and $g = g_{\mathcal{G}_g}$ and since each principal K-bundle may equivalently be described by a cocycle, the set of equivalence classes of principal K-bundles over X is parametrised by

$$\mathrm{Bun}(X, K) = \{\mathcal{K} : \mathcal{K} \text{ is a } K\text{-valued cocycle on } X\} / \sim,$$

where $\sim$ is the equivalence of cocycles described above. ■

Definition B.1.10 (Smooth principal bundle). Let K be a locally convex Lie group and M be a manifold with corners. A continuous principal K-bundle over M is called a *smooth principal K-bundle over M* if P is a manifold with corners and the local trivialisations from (B.1) can be chosen to be diffeomorphisms. A morphism of smooth principal bundles is a morphism of continuous bundles that is also smooth. ■

Remark B.1.11 (Continuous vs. smooth principal bundles). All the remarks on the equivalent description of sections and local trivialisations, principal bundles and cocycles and bundle equivalences remain valid in exactly the same way if one only substitutes the assumptions of being continuous with those of being smooth. In particular, we have the same notions of trivialising subsets, smooth bundle equivalences and smooth sections defining smooth local trivialisations. Smooth local sections in turn define smooth transition functions , cocycles $k_{ij} : U_i \cap U_j \to K$ and bundle equivalences are defined by smooth mappings $g_j : V_J \to K$.

Furthermore, if $\mathcal{P}$ is a smooth principal K-bundle over M, then a *smooth open trivialising system* $\mathcal{U}$ of $\mathcal{P}$ consists of an open cover $(U_i)_{i\in I}$ and smooth sections $\sigma_i : U_i \to P$. If each $\overline{U}_i$ is also a manifold with corners and the section σ_i can be extended to smooth sections $\sigma_i : \overline{U}_i \to P$, then $\overline{\mathcal{U}} = (\overline{U}_i, \sigma_i)_{i\in I}$ is called a *smooth closed trivialising system* of $\mathcal{P}$. In this case, $\mathcal{U}$ is called the trivialising system *underlying* $\overline{\mathcal{U}}$. ■

Remark B.1.12 (Smooth Structure on Smooth Principal Bundles). Let K be a Lie group and $\mathcal{P}$ be a continuous principal K-bundle over the manifold with corners M. If there exists a trivialising cover $(U_i)_{i\in I}$ and trivialisations $\Theta_i : \pi^{-1}(U_i) \to U_i \times K$ such that the corresponding transition functions $k_{ij} : U_i \cap U_j \to K$ are smooth, then we define on P the structure of a manifold with corners by requiring the local trivialisations

$$\Theta_i : \pi^{-1}(U_i) \to U_i \times K$$

to be diffeomorphisms. This actually defines a smooth structure on P, because it is covered by $(\pi^{-1}(U_i))_{i\in I}$ and since the coordinate changes

$$(U_i \cap U_j) \times K \to (U_i \cap U_j) \times K, \quad (x,k) \mapsto \Theta_j(\Theta_i^{-1}(x,k)) = (x, k_{ij}(x)\cdot k)$$

are smooth. ■

Lemma B.1.13 (Existence of smooth trivialising systems). *If $\mathcal{P} = (K, \pi : P \to M)$ is a smooth K-principal bundle with finite-dimensional base M, then there exists an open cover $(V_i)_{i\in I}$ such that each $\overline{V_i}$ is trivialising and a manifold with corners. In particular, there exists a smooth closed trivialising system $\overline{\mathcal{V}} = (\overline{V_i}, \sigma_i)_{i\in I}$, where σ_i is the restriction of some smooth section, defined on an open neighbourhood of $\overline{V_i}$. If, moreover, M is compact then we may assume I to be finite.*

Proof. For each $m \in M$ there exists an open neighbourhood U and a chart $\varphi : U \to (\mathbb{R}^n)^+$ such that U is trivialising, i.e. there exists a smooth section $\sigma : U \to P$. Then there exists an $\varepsilon > 0$ such that $(\mathbb{R}^n)^+ \cap (\varphi(m) + [-\varepsilon, \varepsilon]^n) \subseteq \varphi(U)$ is a manifold with corners and we set $V_m := \varphi^{-1}((\mathbb{R}^n)^+ \cap (\varphi(m) + (-\varepsilon, \varepsilon)^n))$. Then $(V_m)_{m\in M}$ has the desired properties and if M is compact it has a finite subcover. ■

Definition B.1.14 (Associated bundles). Let $\mathcal{P}$ be a smooth principal K-bundle and $\lambda : K \times N \to N$ be a smooth left action of K on some smooth locally convex manifold N. Then we define the *associated bundle* $\lambda(\mathcal{P}) := \mathcal{P} \times_\lambda N$ to consist of the topological space

$$(P \times N)/K,$$

where K acts on $P \times N$ from the right by $(p,n)\cdot k := (p\cdot k, \lambda(k^{-1}, n))$ and the bundle projection

$$\pi_{\lambda(\mathcal{P})} : P \times N \to M, \quad [p,n] \mapsto \pi_{\mathcal{P}}(p),$$

where $\pi_{\mathcal{P}} : P \to M$ is the bundle projection of $\mathcal{P}$. ■

Remark B.1.15 (Local trivialisations in associated bundles). If $\lambda(\mathcal{P}) = \mathcal{P} \times_\lambda N$ is an associated bundle, then it is in particular a locally trivial K-bundle over M with fibre N, i.e., we have for each $m \in M$ an open neighbourhood U, called *trivialising neighbourhood* and a diffeomorphism

$$\Theta : \pi^{-1}_{\lambda(\mathcal{P})}(U) \to U \times N$$

such that for two trivialising neighbourhoods U and U' with local trivialisations Θ and Θ' we have

$$\Theta'(\Theta^{-1}(x,n)) = k_{U'U}(x)^{-1}.n \tag{B.6}$$

for $x \in U \cap U'$ and some smooth function $k_{UU'} : U \cap U' \to K$. In fact, if $\pi_{\mathcal{P}} : P \to M$ is the bundle projection of $\mathcal{P}$, U is a trivialising neighbourhood for $\mathcal{P}$ and $\sigma : U \to \mathcal{P}$ a smooth section of $\pi_{\mathcal{P}}$, then

$$\pi^{-1}_{\lambda(\mathcal{P})}(U) = (U \times N)/K \to U \times N, \quad (p,n) \mapsto (\pi_{\mathcal{P}}(p), k_\sigma(p).n)$$

defines such a diffeomorphism with inverse $(x,n) \mapsto [(\sigma(x),n)]$. Furthermore, two such trivialising neighbourhoods define by (B.2) a smooth map $k_{UU'} : U \cap U' \to K$ such that (B.6) holds. ■

B.2 Classification results for principal bundles

This section provides some results from the classification theory of continuous principal bundles. We focus mostly on bundles over spheres and surfaces, since these are the cases dealt with in Chapter 4.

When treating universal bundles, we will restrict to the case of bundles universal for bundles over CW-complexes. This will suffice, because we are always interested in principal bundles over finite-dimensional manifolds, which are locally finite CW-complexes.

To avoid confusion with the homotopy groups, we denote throughout this chapter the bundle projection with η instead of π.

Definition B.2.1 (Pull-back bundle). If $\mathcal{P}$ is a continuous (respectively smooth) principal K-bundle over M and $f : N \to M$ is a continuous (respectively smooth) map, then $f^*(\mathcal{P}) = (K, f^*(\eta) : f^*(P) \to N)$ is the *pull-back bundle*, where

$$f^*(P) = \{(n,p) \in N \times P : f(n) = \eta(p)\}$$

and $f^*(\eta)(n,p) = n$. Furthermore, we have an action

$$f^*(\rho) : f^*(P) \times K \to f^*(P), \quad (n,p) \cdot k = (n, p \cdot k)$$

and an induced map $f_{\mathcal{P}} : f^*(P) \to P$, $(n,p) \mapsto p$. ■

Lemma B.2.2 (Cocycle for pull-back bundle). *If $\mathcal{P}$ is a continuous (respectively smooth) principal K-bundle, then $f^*(\mathcal{P})$ is a continuous (respectively smooth) principal bundle, and $f_{\mathcal{P}}$ is a continuous (respectively smooth) bundle map.*

Furthermore, if $\mathcal{K} = (k_{ij} : U_i \cap U_j \to K)_{i,j\in I}$ is a cocycle describing $\mathcal{P}$, then $f^*(k_{ij}) : f^{-1}(U_i \cap U_j) \to K$, $n \mapsto k_{ij}(f(n))$ are the transition functions of a cocycle $f^*(\mathcal{K})$ of $f^*(\mathcal{P})$.

Proof. If $(U_i)_{i\in I}$ is the open cover underlying $\mathcal{K}$, then $(f^{-1}(U_i))_{i\in I}$ is an open cover of N. Furthermore, if $\sigma_i : U_i \to P$ is a section of $\mathcal{P}$ with corresponding $k_{\sigma_i} : \eta^{-1}(U_i) \to K$, then $f^*(\sigma_i) : f^{-1}(U_i) \to f^*(P)$, $n \mapsto (n, \sigma_i(f(n)))$ is a section of $f^*(\mathcal{P})$, and

$$f^*(\eta)^{-1}(f^{-1}(U_i)) \to f^{-1}(U_i) \times K, \quad (n,p) \mapsto (n, k_i(p))$$

defines local trivialisations of $f^*(\mathcal{P})$ with $f^*(k_{ij})$ as transition functions. ■

Definition B.2.3 (Universal bundle). Let $\mathcal{P}_K = (K, \eta_K : EK \to BK)$ be a continuous principal K-bundle for a topological group K. Then $\mathcal{P}_K$ is called a *universal bundle* and BK is called a *classifying space* for K if for each other continuous principal K-bundle $\mathcal{P} = (K, \eta : P \to X)$ over a CW-complex X, there exists a map $c : X \to BK$, called *classifying map*, such that $c^*(\mathcal{P}_K)$ is equivalent to $\mathcal{P}$, and, furthermore, if two maps $c : X \to BK$ and $c' \to BK$ are homotopic if and only if $f^*(\mathcal{P}_K)$ and $f'^*(\mathcal{P}_K)$ are equivalent.

In other words, $\mathcal{P}_K$ is universal if for each CW-complex X the map

$$[X, BG]_* \to \mathrm{Bun}(X, K), \quad [f] \mapsto [f^*(EK)],$$

where the brackets around f denote the homotopy class of f and around $f^*(EK)$ the equivalence class of $f^*(EK)$, is well-defined and a bijection. ■

Theorem B.2.4 (Existence of universal bundles). *([Mi56]) If K is a topological group, then there exists a continuous principal K-bundle $\mathcal{P}_K$ which is universal.* ■

Corollary B.2.5 (Bundles over contractible spaces are trivial). *A continuous principal K-bundle $\mathcal{P}$ over a contractible CW-complex X is necessarily trivial.*

Proof. If X is contractible, then each classifying map is homotopic to a constant map and the pull-back bundle of a constant map is trivial. ■

Theorem B.2.6 (Criterion for universal bundle). *([Hu94, Theorem 13.1]) If $\mathcal{P}$ is a continuous principal K-bundle, then $\mathcal{P}$ is universal if and only if $\pi_n(P) = 0$ for all $n \in \mathbb{N}_0$.* ■

Corollary B.2.7 (Homotopy groups of classifying spaces). *If $\mathcal{P}_K$ is a universal continuous principal K-bundle, then $\pi_{n+1}(BK) \cong \pi_n(K)$ for all $n \in \mathbb{N}_0$.*

Proof. Since a locally trivial bundle is in particular a Serre fibration [Br93, Corollary VII.6.12], this is an immediate consequence of the long exact homotopy sequence [Br93, Theorem VII.6.7] and Theorem B.2.6. ∎

Proposition B.2.8 (Classification of bundles over Spheres). *The set of equivalence classes of continuous principal K-bundles over $\mathbb{S}^m$ is parametrised by $\pi_{m-1}(K)$.*

Proof. This follows from $\mathrm{Bun}(\mathbb{S}^m, K) \cong [\mathbb{S}^m, BK]_* \cong \pi_m(BK) \cong \pi_{m-1}(K)$. ∎

Remark B.2.9 (Description of bundles over spheres). The bijection from Proposition B.2.10 can be obtained as follows. Identify $\mathbb{S}^n$ with $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. Then

$$U_N := \{x \in \mathbb{S}^n : x_{n+1} \geq 0\} \text{ and } U_S := \{x \in \mathbb{S}^n : x_{n+1} \leq 0\}$$

are the northern and southern hemisphere with north pole $x_N = (0, \ldots, 0, 1)$ and south pole $x_S = (0, \ldots, 0, -1)$ and we have

$$U_N \cap U_S = \mathbb{S}^n \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\} \cong \mathbb{S}^{n-1}.$$

We will assume that x_S is the base-point of $\mathbb{S}^n$. Furthermore, if $\mathcal{P}$ is a continuous principal K-bundle over $\mathbb{S}^n$, then there exist sections $\sigma_N : U_N \to P$ and $\sigma_S : U_S \to P$, because U_N and U_S are contractible (cf. Corollary B.2.5). If $\sigma_S(x_S) \cdot k = p_0$, then $x \mapsto \sigma_S(x) \cdot k$ defines a new base-point preserving section. In the same way, if x_0 is the base-point of $\mathbb{S}^{n-1} \cong U_N \cap U_S$, and $\sigma_N(x_0) \cdot k' = \sigma_S(x_0)$, then $\sigma'_N(x) := \sigma_N(x) \cdot k'$ defines a section that coincides with σ_S in x_0. Then

$$\sigma'_N(x) = \sigma_S(x) \cdot c_{\mathcal{P}}(x) \;\text{ if }\; x \in U_N \cap U_S \cong \mathbb{S}^{n-1}$$

defines $c_{\mathcal{P}} \in C_*(\mathbb{S}^{n-1}, K)$, and we may take $[c_{\mathcal{P}}]$ as a representative in

$$\mathrm{Bun}(\mathbb{S}^n, K) \cong [\mathbb{S}^n, BK]_* \cong \pi_{n-1}(K).$$

Since $c : \mathbb{S}^{n-1} \to K$ and $c' : \mathbb{S}^{n-1} \to K$ are homotopic if and only if $c \cdot c^{-1} : \mathbb{S}^{n-1} \cong \partial\mathbb{B}^n \to K$ extends to $\mathbb{B}^n$, it follows with Remark B.1.9 that $[\mathcal{P}] \mapsto [c_{\mathcal{P}}]$ is actually bijective. In particular, principal K-bundles over $\mathbb{S}^1$ are (up to equivalence) of the following form. For $k \in K$ denote

$$P_k := \mathbb{R} \times K/\sim \quad \text{with} \quad (x, k') \sim (x + n, k^{-n} \cdot k').$$

Then K acts naturally on P_k by $[(x, k') \cdot k''] = [(x, x' \cdot k'')]$ and $\eta : P_k \to \mathbb{S}^1$, $[(x, k')] \mapsto [x]$ is a bundle projection, where we identified $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$. The above considerations show that $\mathcal{P}_k$ is classified by $[k] \in \pi_0(K)$. Furthermore, if K is a Lie group, then $\mathcal{P}_k$ is also a smooth principal K-bundle, since there exists a trivialising system such that the transition functions take values in $\{e, k\}$ and thus are smooth.

Alternatively, a representing map $c \in C_*(\mathbb{S}^{n-1}, K)$ can also be obtained as follows. We consider $\mathbb{S}^m$ as the quotient $\mathbb{B}^n/\partial\mathbb{B}^n$ and denote by $q : \mathbb{B}^n \to \mathbb{S}^n$ the corresponding quotient map. Then there exists a map $Q : \mathbb{B}^n \to P$ with $\eta \circ Q = q$, since $\eta : P \to \mathbb{S}^n$ is a locally trivial bundle and thus a Serre fibration (cf. [Br93, Corollary VII.6.12]). Then $Q(\partial\mathbb{B}^m) \subseteq \eta^{-1}(x_S)$ and thus $Q(x_0) = Q(x) \cdot c(x)$ for $x \in \partial\mathbb{B}^n$, where x_0 is the base-point of $\partial\mathbb{B}^m \cong \mathbb{S}^{n-1}$. Since $c(x) = k_\sigma(Q(x_0)) \cdot k_\sigma(Q(x))^{-1}$ for any section $\sigma : U_S \to P$ defining $k_\sigma : \eta^{-1}(U_S) \to P$ by $p = \sigma(\eta(p)) \cdot k_\sigma(p)$, we furthermore have that c is continuous and thus $c \in C_*(\mathbb{S}^{n-1}, K)$.

Since $q|_{B_{\frac{1}{2}}(0)}$ is a homeomorphism onto U_N, the map $Q|_{B_{\frac{1}{2}}(0)}$ determines a section σ_N on U_N. Setting $\sigma_S(x) = Q(x) \cdot c(x \cdot \|x\|^{-1})^{-1}$, this defines a continuous map on $\mathbb{B}^m \setminus \operatorname{int}(B_{\frac{1}{2}(0)})$ which is constant on $\partial\mathbb{B}^n$ and thus a section on U_S. For $x \in \partial B_{\frac{1}{2}(0)}$ we have $\sigma_N(x) = \sigma_S(x) \cdot c(2x)$, and thus c also represents $\mathcal{P}$ in $\operatorname{Bun}(\mathbb{S}^n, K) \cong \pi_{n-1}(K)$. ■

Proposition B.2.10 (Classification of bundles over surfaces). *Let K be a connected topological group and Σ be an oriented surface. Then* $\operatorname{Bun}(\Sigma, K) \cong \pi_1(K)$ *if Σ is closed and compact and* $\operatorname{Bun}(\Sigma, K)$ *is trivial if Σ is non-compact or non-closed.*

Proof. Since $\pi_0(K) \cong \pi_1(BK)$ by Corollary B.2.7 and $H^i(\Sigma) = 0$ for $i > 2$, we have

$$\operatorname{Bun}(\Sigma, K) \cong [\Sigma, BK]_* \cong H^2(\Sigma, \pi_2(BK))$$

by [Br93, Corollary VII.13.16]. Since $H_1(\Sigma)$ is free, [Br93, Corollary V.7.2] now yields

$$H^2(\Sigma, \pi_2(BK)) \cong \operatorname{Hom}(H_2(\Sigma), \pi_2(BK)) \cong \operatorname{Hom}(H_2(\Sigma), \pi_1(K)),$$

and the assertion follows from $H_2(\Sigma) \cong \mathbb{Z}$ in the case of a compact and closed surface and $H_2(\Sigma) \cong 0$ otherwise. ■

Remark B.2.11 (Notation for surfaces). We recall some facts on the classification of compact surfaces (cf. [Ma67, Theorem 5.1], [Ne02b, Remark IV.4.5]). Each closed compact orientable surface Σ of genus g can be described as a CW-complex by starting with a *bouquet*

$$A_{2g} = \underbrace{\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1}_{2g}$$

of $2g$ circles. Denote by $\mathbb{S}^1_i$ the i-th circle in this bouquet. We write $a_1, \ldots, a_{2g}$ for the corresponding generators of the fundamental group of A_{2g}, which is a free group on $2g$ generators, and represent a_i by the inclusion $\alpha_i : \mathbb{S}^1 \hookrightarrow \mathbb{S}^1_i \subseteq \mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1$. Then we consider a continuous map $f_\Sigma : \mathbb{S}^1 \to A_{2g}$ representing

$$a_1 \cdot a_2 \cdot a_1^{-1} \cdot a_2^{-1} \cdots a_{2g-1} \cdot a_{2g} \cdot a_{2g-1}^{-1} \cdot a_{2g}^{-1} \in \pi_1(A_{2g}). \tag{B.7}$$

Now Σ is homeomorphic to the space obtained by identifying the points on $\partial\mathbb{B}^2 \cong \mathbb{S}^1$ with their images in A_{2g} under f_Σ, i.e.,

$$\Sigma \cong A_{2g} \cup_{f_\Sigma} \mathbb{B}^2, \tag{B.8}$$

and we denote by q_Σ the corresponding quotient map $q_\Sigma : \mathbb{B}^2 \to \Sigma$. Thus we can identify A_{2g} with the subset $A_{2g} = \Sigma \setminus \operatorname{int}(\mathbb{B}^2)$ of Σ, $\operatorname{int}(\mathbb{B}^2)$ is itself a subset of Σ and we take the base-point of A_{2g} as base-point of Σ. Furthermore, note that with respect to this identification we have $\Sigma/A_{2g} \cong \mathbb{S}^2$ and we denote by $q_{\mathbb{S}^2} : \Sigma \to \mathbb{S}^2$ the corresponding quotient map.

The most instructive picture is to view $\mathbb{B}^2$ as a regular polygon with $4g$ edges, where we identify certain points on the edges such that in counterclockwise order the sequence of edges corresponds to the loop

$$a_1 a_2 a_1^{-1} a_2^{-1} \ldots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1}.$$

Now Σ corresponds to the polygon modulo these identifications. ■

Remark B.2.12 (Description of bundles over surfaces). Let Σ be a compact, closed and orientable surface and K be a topological group. The bijection $\operatorname{Bun}(\Sigma, K) \cong \pi_1(K)$ from Proposition B.2.10 can be obtained as follows.

At first, we obtain a map $\operatorname{Bun}(\mathbb{S}^2, K) \to \operatorname{Bun}(\Sigma, K)$ as follows. Let $q : \Sigma \to \mathbb{B}^2$ be the quotient map identifying A_{2g} with the base-point in $\mathbb{S}^2$ (cf. Remark B.2.11). For each continuous bundle $\mathcal{P}_{\mathbb{S}^2}$ over $\mathbb{S}^2$ we have the corresponding pull-back bundle $\mathcal{P}_\Sigma = q^*(\mathcal{P}_{\mathbb{S}^2})$ given by

$$\begin{array}{ccc} P_\Sigma & \xrightarrow{Q} & P_{\mathbb{S}^2} \\ {\scriptstyle \eta_\Sigma}\downarrow & & {\scriptstyle \eta_{\mathbb{S}^2}}\downarrow \\ \Sigma & \xrightarrow{q} & \mathbb{S}^2. \end{array}$$

If $c : \mathbb{S}^2 \to BK$ is a classifying map for $\mathcal{P}_{\mathbb{S}^2}$, then $c \circ q$ is a classifying map for $\mathcal{P}_\Sigma$. Furthermore if $F : [0,1] \times \mathbb{S}^2 \to BK$ is a homotopy, then $F \circ (\mathrm{id}_{[0,1]} \times q) : [0,1] \times \Sigma \to BK$ is a homotopy and we thus obtain a well-defined map

$$\operatorname{Bun}(\mathbb{S}^2, K) \to \operatorname{Bun}(\Sigma, K), \quad [\mathcal{P}_{\mathbb{S}^2}] \mapsto [\mathcal{P}_\Sigma]. \tag{B.9}$$

Since K is assumed to be connected, BK is simply connected (cf. Corollary B.2.7) and thus each map $A_{2g} \to BK$ is homotopic to a constant map. This in turn implies that a classifying map $c_\Sigma : \Sigma \to BK$ can always be chosen to be constant on A_{2g} and thus factors through a map $c_{\mathbb{S}^2} : \mathbb{S}^2 \to BK$. This shows that (B.9) is surjective. The same argument shows that (B.9) is also injective and thus provides a bijection $\operatorname{Bun}(\Sigma, K) \cong \operatorname{Bun}(\mathbb{S}^2, K) \cong \pi_1(K)$. ■

Proposition B.2.13 (Bundles over 3-dimensional manifolds). *If K is a simply connected finite-dimensional Lie group, then any continuous principal K-bundle over a 3-dimensional manifold is trivial.*

Proof. With $\pi_3(BK) \cong \pi_2(K) \cong 0$, $\pi_2(BK) \cong \pi_1(K) \cong 0$, $\pi_1(BK) \cong \pi_0(K) \cong 0$ this follows as in Proposition B.2.10. ■

Remark B.2.14 (Bundles whose structure group is a $\boldsymbol{K(n,G)}$). If X is a topological space with non-trivial $G = \pi_n(X)$ for all but one $n \in \mathbb{N}$, then it is called an *Eilenberg–MacLane space* $K(n,G)$. In particular, if a topological group K is a $K(n,G)$, then BK is a $K(n+1,G)$ and [Br93, Corollary VII.13.16] implies

$$[X, BK]_* \cong H^{n+1}(X,G)$$

Since $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a $K(1,\mathbb{Z})$, this shows that for any X, $\operatorname{Bun}(X, \mathbb{S}^1) \cong H^2(X,\mathbb{Z})$. Furthermore, if $\mathcal{H}$ is an separable infinite-dimensional Hilbert space, then $U(\mathcal{H})$ is contractible (cf. [Ku65]) and $Z(U(\mathcal{H})) \cong U(1)$ implies that $\operatorname{PU}(\mathcal{H}) = U(\mathcal{H})/Z(U(\mathcal{H}))$ is a $K(2,\mathbb{Z})$ and thus

$$\operatorname{Bun}(X, \operatorname{PU}(\mathcal{H})) \cong [X, B\operatorname{PU}(\mathcal{H})]_* \cong H^3(X,\mathbb{Z}). \qquad ■$$

B.3 Connections on principal bundles

Connections describe the geometric aspects of smooth principal and vector bundles. In this section we give the basic definition and relate these two concepts.

Definition B.3.1 (Vertical invariant vector fields). If $\mathcal{P}$ is a smooth principal K-bundle, then $V_p := \ker(T\pi(p)) \subseteq T_pP$ is called the *vertical tangent space*. Furthermore, if $\mathcal{V}(P)^K$ denotes the subspace of $\mathcal{V}(P)$ satisfying $T\rho_k \circ X = X \circ \rho_k$ for all $k \in K$, then the space of *vertical K-invariant vector fields* is the closed subspace

$$\mathcal{V}_{\text{vert}}(P)^K := \{X \in \mathcal{V}(P)^K : X(p) \in V_p \text{ for all } p \in P\}. \qquad ■$$

Lemma B.3.2 (Isomorphism to the gauge algebra). *If $\mathcal{P}$ is a smooth principal K-bundle and if we consider the fibre $\pi^{-1}\big(\pi(p)\big)$ as a sub-manifold of P, then $V_p \cong T_p\pi^{-1}\big(\pi(p)\big)$. Each V_p is canonically isomorphic as a vector space to the Lie algebra $\mathfrak{k}$, where the isomorphism is given by*

$$\tau_p : \mathfrak{k} \to T_pP, \quad x \mapsto d\rho_p(e).x,$$

where $\rho_p : K \to P$, $k \mapsto p \cdot k$ is the orbit map *at $p \in P$. Furthermore, we have a canonical $C^\infty(M,\mathbb{R})$-linear isomorphism of topological Lie algebras*

$$\iota : C^\infty(P,\mathfrak{k})^K \to \mathcal{V}_{\text{vert}}(P)^K, \quad \iota(\eta)(p) = -\tau_p(\eta(p))$$

and thus a closed $C^\infty(M,\mathbb{R})$-linear embedding $C^\infty(P,\mathfrak{k})^K \xhookrightarrow{\iota} \mathcal{V}(P)^K$.

Proof. The first assertion follows from the fact that $\pi^{-1}(\pi(p))$ is diffeomorphic to K. It suffices to check the second in local trivialisations, so let U be a trivialising neighbourhood of p with section $\sigma : U \to P$ and corresponding $k_\sigma : \pi^{-1}(U) \to K$. Then

$$\delta^l(k_\sigma)(p)\big|_{V_p} : \mathcal{V}_{\mathrm{vert}}(P)^K \to C^\infty(\pi^{-1}(U), \mathfrak{k})^K$$

defines a continuous inverse of ι. ■

Definition B.3.3 (Connection 1-forms). Let $\mathcal{P}$ be a smooth principal K-bundle over the finite-dimensional closed manifold M. Then a connection on $\mathcal{P}$ is given by a connection 1-form $A \in \Omega^1(P, \mathfrak{k})$ satisfying

$$\tau_p \circ A|_{V_p} = \mathrm{id}_{V_p} \quad \text{for all} \quad p \in P \qquad \text{and} \qquad A \circ T\rho_k = \mathrm{Ad}(k^{-1}).A \quad \text{for all} \quad k \in K.$$

■

Remark B.3.4 (Connections as horizontal lift of vector fields). A connection 1-form on $\mathcal{P}$ determines a $C^\infty(M, \mathbb{R})$-linear splitting of the exact sequence

$$C^\infty(P, \mathfrak{k})^K \xhookrightarrow{\iota} \mathcal{V}(P)^K \xrightarrow{T\pi_*} \mathcal{V}(M),$$

where $T\pi_*(X)(m) = T\pi(X(p))$ for some $p \in \pi^{-1}(m)$. In fact,

$$\mathcal{V}(P)^K \ni X \mapsto A.X \in C^\infty(P, \mathfrak{k})^K$$

defines a continuous inverse to ι and thus a splitting. Then the *corresponding horizontal* lift

$$S : \mathcal{V}(M) \to \mathcal{V}(P)^K, \quad X \mapsto S(X)$$

is given by $X \mapsto \widetilde{X} - A.\widetilde{X}$ for an arbitrary lift $\widetilde{X}$ of X. ■

Remark B.3.5 (Isomorphisms of sections and invariant mappings). Let $\mathcal{P}$ be a smooth principal K-bundle, $\lambda : K \times Y \to Y$ be a smooth action of K on the locally convex space Y, and let $\lambda(\mathcal{P})$ be the corresponding associated smooth vector bundle. Then the space of sections $S(\lambda(\mathcal{P})) = \Omega^0(M, \lambda(\mathcal{P}))$ is isomorphic to

$$C^\infty(P, Y)^\lambda := \{f \in C^\infty(P, Y) : f(p \cdot k) = \lambda(k^{-1}, f(p))\},$$

where the isomorphism is given by $C^\infty(P, Y)^\lambda \to S(\lambda(P))$, $f \mapsto \sigma_f$ with

$$\sigma_f(m) = [p, f(p)] = [p \cdot k^{-1}, \lambda(k, f(p))] = [p \cdot k^{-1}, f(p \cdot k^{-1})].$$

Furthermore, if

$$\Omega^1_{\mathrm{bas}}(P, Y)^\lambda := \{\omega \in \Omega^1(P, Y) : \omega \circ T\rho_k = \mathrm{Ad}(k^{-1}).\omega,\ \omega|_{V_p} \equiv 0\ \forall k \in K, p \in P\}$$

denotes the space of *based invariant* 1-forms on $\lambda(\mathcal{P})$, then $\Omega^1_{\mathrm{bas}}(P,Y)^\lambda \cong \Omega^1(M,\lambda(\mathcal{P}))$, where the isomorphism is given by

$$\Omega^1_{\mathrm{bas}}(P,Y)^\lambda \to \Omega^1(M,\lambda(\mathcal{P})), \quad \omega \mapsto \omega_M$$

with $\omega_M(X_m) = [p, \omega(\widetilde{X}_p)]$, where $\widetilde{X}_p \in T_pP$ is such that $T\pi(\widetilde{X}_p) = X_m$. Note that this is well-defined, because for $\widetilde{X}_p'$ with $T\pi(\widetilde{X}_p') = X_m$ we have $\widetilde{X}_p - \widetilde{X}_p' \in V_p$, which implies that $\omega(X_m)$ does not depend on the choice of $\widetilde{X}_p$ in T_pP. Furthermore, $[p, \omega(\widetilde{X}_p)] = [p \cdot k, \mathrm{Ad}(k)^{-1}.\omega(\widetilde{X}_p)] = [p \cdot k, \omega(T\rho_k(\widetilde{X}_p))]$ implies that $\omega_M(X_m)$ does not depend on the choice of p. ■

Definition B.3.6 (Covariant derivative). If $\mathcal{E}$ is a smooth vector bundle over the finite-dimensional manifold M without boundary, then a *covariant derivative* is a continuous linear map

$$\nabla : \Omega^0(M,\mathcal{E}) \to \Omega^1(M,\mathcal{E})$$

such that $\nabla(f \cdot \omega).X = (df.X) \cdot \omega + f \cdot (\nabla(\omega).X)$ for all $f \in C^\infty(M)$, $\omega \in \Omega^0(M,\mathcal{E})$ and $X \in \mathcal{V}(M)$. If $\omega \in \Omega^0(M,\mathcal{E})$, then we write shortly $\nabla\omega$ for $\nabla(\omega)$. ■

Lemma B.3.7 (Connection 1-forms inducing covariant derivatives). *Let $\mathcal{P}$ be a smooth principal K-bundle and $\lambda : K \times Y \to Y$ be a smooth action of K. Then a connection 1-form A induces a continuous map*

$$d^A : C^\infty(P,Y)^\lambda \to \Omega^1(P,Y), \quad d^A(\eta)(X_p) = d\eta(X_p) - \dot{\lambda}(A(X_p), \eta(p)).$$

Furthermore, d^A takes values in $\Omega^1_{\mathrm{bas}}(P,Y)^\lambda$ and determines a covariant derivative with respect to the identifications $\Omega^0(M,\lambda(\mathcal{P})) \cong S(\lambda(\mathcal{P})) \cong C^\infty(P,Y)^\lambda$ and $\Omega^1(M,\lambda(\mathcal{P})) \cong \Omega^1_{\mathrm{bas}}(P,Y)^\lambda$.

Proof. Since d^A is given locally in terms of push-forwards of continuous mappings, it is continuous. Since each $X_p \in V_p$ can be written as $\tau_p(x) = d\rho_p(e).x$ for some $x \in \mathfrak{k}$ with $\rho_p : K \to P$, $k \mapsto p \cdot k$, we have

$$\begin{aligned} d^A\,\eta(X_p) &= d\eta(\tau_p(x)) - \dot{\lambda}(A(\tau_p(x)), \eta(p)) = d\eta(d\rho_p(e).x) - \dot{\lambda}(x, \eta(p)) \\ &= d(\eta \circ \rho_p)(e).x - \dot{\lambda}(x), \eta(p)) = d(\lambda(\cdot, \eta(p)))(e).x - \dot{\lambda}(x, \eta(p)) = 0. \end{aligned}$$

Thus d^A actually takes values in $\Omega^1_{\mathrm{bas}}(P,Y)^\lambda$. It is clear that d^A is linear, and because $d(f \cdot \eta)(X_p) = df(X_p) \cdot \eta(p) + f(p) \cdot d\eta(X_p)$, it defines a covariant derivative. ■

Remark B.3.8 (Covariant derivative induced from horizontal lift). If A is a connection 1-from on $\mathcal{P}$ and $S : \mathcal{V}(M) \to \mathcal{V}(P)^K$ is the corresponding lift from Remark B.3.4, then we obtain the covariant derivative also by

$$d^A : C^\infty(P,Y)^\lambda \to \Omega^1_{\mathrm{bas}}(P,Y)^\lambda, \eta \mapsto S(X).\eta$$

with respect to the identifications $\Omega^0(M,\lambda(\mathcal{P})) \cong C^\infty(P,Y)^\lambda$ and $\Omega^1(M,\lambda(\mathcal{P})) \cong \Omega^1_{\mathrm{bas}}(P,Y)^\lambda$. ■

Remark B.3.9 (Invariant forms inducing fibrewise bilinear forms). Let $\mathcal{P}_1$ be a smooth K_1-principal bundle over M, K_2 be a Lie group and $\varphi : K_1 \to K_2$ be a morphism of Lie groups. Then φ induces a smooth principal K_2-bundle over M by composing the transition functions of a cocycle representing $\mathcal{P}_1$ with φ. Furthermore, we have a map $\Phi : P_1 \to P_2$, which is locally given by $(m,k) \mapsto (m, \varphi(k))$ which satisfies $\Phi(p\cdot k) = \Phi(p)\cdot\varphi(k)$.

Now let $\lambda_1 : K_1 \times Y_1 \to Y_1$ and $\lambda_2 : K_2 \times Y_2 \to Y_2$ be smooth actions of K_1 and K_2. Then the two associated vector bundles $\lambda_1(\mathcal{P}_1) = (Y_1, \xi_1 : P_1 \to M)$ and $\lambda_2(\mathcal{P}_2) = (Y_2, \xi_2 : P_2 \to M)$ are given by

$$P_{\lambda_1} = P_1 \times_{\lambda_1} Y_1 = P_1 \times Y_1/\sim \quad \text{with} \quad (p,x) \sim (p\cdot k, \lambda_1(k^{-1}).x),$$
$$P_{\lambda_2} = P_2 \times_{\lambda_2} Y_2 = P_2 \times Y_2/\sim \quad \text{with} \quad (p,x) \sim (p\cdot k, \lambda_2(k^{-1}).x).$$

Furthermore, let $\kappa : Y_1 \times Y_1 \to Y_2$ be continuous, bilinear and φ-equivariant map, i.e., $\kappa(\lambda_1(k).x, \lambda_1(k).x') = \lambda_2(\varphi(k)).\kappa(x,x')$ for all $x,x' \in Y$ and $k \in K$. For $p,p' \in P_1$ we define $k_{p^{-1}p'} \in K_1$ by $p = p' \cdot k_{p^{-1}p'}$, whence $k_{(p\cdot k)^{-1}(p'\cdot k')} = k^{-1}\cdot k_{p^{-1}p'} \cdot k'$ and $k_{p^{-1}p'} = e$ if $p = p'$. If $p,p' \in P_1$, $k,k' \in K_1$ and $x,x' \in Y_1$, then we have

$$\begin{aligned}\Big[\Phi(p\cdot k), \kappa\big(\lambda_1(k^{-1}).x, \lambda_1(k^{-1}\cdot k_{p^{-1}p'}\cdot k').\lambda(k'^{-1}).x'\big)\Big]& \\ = \big[\Phi(p)\cdot\varphi(k), \lambda_2(\varphi(k^{-1})).\kappa(x, \lambda(k_{p^{-1}p'}).x')\big].& \end{aligned} \tag{B.10}$$

Thus we can fibrewise define bilinear maps

$$\kappa(\cdot,\cdot)_m : E_m \times E_m \to F_m, \quad \kappa([p,x],[p',x'])_{\pi(p)} := [\Phi(p), \kappa(x, \lambda_2(k_{p^{-1}p'}).x')],$$

where $E_m = \xi_1^{-1}(m)$, $F_m = \xi_2^{-1}(m)$ are the corresponding fibres over m and. That this is in fact well-defined follows from (B.10). In particular, if $K_1 = K_2 = K$, $\varphi = \mathrm{id}_K$, $\lambda_1 = \lambda_2 = \mathrm{Ad}$ and κ is the Lie bracket $[\cdot,\cdot]_{\mathfrak{k}}$, which is K-equivariant for the adjoint action, this construction defines a Lie bracket $[\cdot,\cdot]_m$ on each $(E_{\mathrm{ad}})_m$. ∎

Definition B.3.10 (Multiplication induced from invariant forms). In the situation of Remark B.3.9, we define the multiplication

$$\kappa_* : \Omega^p(M, \lambda_1(P_1)) \times \Omega^q(M, \lambda_1(\mathcal{P}_1)) \to \Omega^{p+q}(M, \lambda_2(\mathcal{P}_2)), \quad (\omega, \omega') \mapsto \kappa_*(\omega,\omega'),$$

where

$$\kappa_*(\omega,\omega')(X_{1,m},\dots,X_{p+q,m}) = \sum_{\sigma\in S_{p+q}} \mathrm{sgn}(\sigma)\kappa(\omega_m(X_{\sigma(1),m},\dots,X_{\sigma(p),m}), \omega'_m(X_{\sigma(p+1),m},\dots,X_{\sigma(p+q)}))_m$$

for $X_{1,m},\dots,X_{p+q,m} \in T_mM$. In particular, if $K_1 = K_2 = K$, $\varphi = \mathrm{id}_K$, $\lambda_1 = \lambda_2 = \mathrm{Ad} : K \times \mathfrak{k} \to \mathfrak{k}$ and κ is the Lie bracket $[\cdot,\cdot]_{\mathfrak{k}}$, then this defines a Lie bracket on the space of sections $S(\mathrm{Ad}(\mathcal{P}))$ by $[\sigma,\sigma'](m) = [\sigma(m), \sigma'(m)]_m$. ∎

Lemma B.3.11 (Continuity of the multiplication). *In the situation of Remark B.3.9, if $(p,q) \in \{(0,0),(1,0),(0,1)\}$, then*

$$\kappa_* : \Omega^p(M, \lambda_1(\mathcal{P}_1)) \times \Omega^q(M, \lambda_1(\mathcal{P}_1)) \to \Omega^{p+q}(M, \lambda_2(\mathcal{P}_2))$$

is continuous.

Proof. This is immediate, since in local coordinates κ_* is given by the push-forward of a continuous map which is continuous by Proposition 2.2.22. ■

Proposition B.3.12 (Sections in adjoint bundle form a Lie algebra). *If $\mathcal{P}$ is a smooth principal K-bundle, then the Lie bracket*

$$[\sigma, \sigma'](m) = [\sigma(m), \sigma'(m)]_m$$

on the space of section $S(\mathrm{Ad}(\mathcal{P}))$ turns $S(\mathrm{Ad}(\mathcal{P}))$ into a locally convex Lie algebra isomorphic to $C^\infty(P, \mathfrak{k})^K$.

Proof. This is an immediate consequence of Remark B.3.5 and Lemma B.3.11. ■

Lemma B.3.13 (Naturality of covariant derivative and multiplication). *Let $\mathcal{P}$ be a smooth principal K-bundle and $\lambda_1 : K \times Y_1 \to Y_1$ and $\lambda_2 : K \times Y_2 \to Y_2$ be two smooth actions of K and $\kappa : Y_1 \times Y_1 \to Y_2$ be K-equivariant (i.e., id_K-equivariant in the sense of Remark B.3.9). If $A \in \Omega^1(P, \mathfrak{k})$ is a connection 1-form on $\mathcal{P}$ and $d^A_{1,2} : \Omega^0(M, \lambda_{1,2}(P)) \to \Omega^1(M, \lambda_{1,2}(P))$ are the corresponding covariant derivatives, then we have for $\eta, \mu \in \Omega^0(M, \lambda_1(\mathcal{P}))$*

$$d_2^A\, \kappa_*(\eta, \mu) = \kappa_*(d_1^A\, \eta, \mu) + \kappa_*(\eta, d_1^A\, \mu).$$

In particular, if $\lambda_1 = \lambda_2 = \mathrm{Ad} : K \times \mathfrak{k} \to \mathfrak{k}$, and κ is the Lie bracket $[\cdot,\cdot]_{\mathfrak{k}}$, then $d_1^A = d_2^A =: d^A$ and we have

$$d^A\, [\eta, \mu] = [d^A\, \eta, \mu] + [\eta, d^A\, \mu].$$

Proof. Since κ is K-invariant, we have $\lambda_2 \circ (\mathrm{id}_K \times \kappa) = \kappa \circ (\lambda_1 \times \lambda_1) \circ \Delta$, with

$$\Delta : K \times Y_1 \times Y_1 \to K \times Y_1 \times K \times Y_1, \quad \Delta(k, y, y') = (k, y, k, y').$$

This implies

$$\begin{aligned}
\dot{\lambda}_2(x, \kappa(y, y')) &= d(\lambda_2 \circ (\mathrm{id}_K \times \kappa))(e, y, y')(x, 0, 0) \\
&= d(\kappa \circ (\lambda_1 \times \lambda_1) \circ \Delta)(e, y, y')(x, 0, 0) \\
&\overset{i)}{=} \kappa(\lambda_1(e, y), d\lambda_1(e, y')(x, 0)) + \kappa(d\lambda_1(e, y)(x, 0), \lambda_1(e, y')) \\
&= \kappa(y, \dot{\lambda}_1(x, y')) + \kappa(\dot{\lambda}_1(x, y), y')
\end{aligned}$$

where $i)$ holds, because κ is bilinear. We thus have

$$\begin{aligned}
d_2^A\, \kappa_*(\eta, \mu) = d\kappa_*(\eta, \mu) - \dot{\lambda}_2(A, \kappa_*(\eta, \mu)) = \kappa_*(d\eta, \mu) + \kappa_*(\eta, d\mu) \\
- \kappa_*(\eta, \dot{\lambda}_1(A, \mu)) - \kappa_*(\dot{\lambda}_1(A, \eta), \mu) = \kappa_*(d_1^A\, \eta, \mu) + \kappa_*(\mu, d_1^A\, \mu).
\end{aligned}$$ ■

Lemma B.3.14 (Canonical connection on bundles over the circle). *Let $\mathcal{P}_k$ be a smooth principal K-bundle over $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, given by some $k \in K$ as in Remark B.2.9. If we identify $\mathcal{V}(\mathbb{S}^1)$ with the $\mathbb{Z}$-invariant vector fields on $\mathbb{R}$ and $\mathfrak{gau}(\mathcal{P})$ with $C^\infty_k(\mathbb{S}^1, \mathfrak{k})$, then there is a connection 1-form on $\mathcal{P}_k$ inducing $f \mapsto df$ as its covariant derivative on $\mathcal{P}_k$.*

Proof. First we note that $f \mapsto df$ defines in fact a covariant derivative, since

$$(df.X)(t+n) = f'(t+n) \cdot X(t+n) = (\mathrm{Ad}(k^{-n}).f'(t)) \cdot X(t) = \mathrm{Ad}(k^{-n}).((df.X)(t)).$$

We may cover $\mathbb{S}^1$ with two arcs U_1, U_2 and choose trivialisations of $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ such that the transition function k_{12} is locally constant. Then the trivialisations define $k_1 : \pi^{-1}(U_1) \to K$ and $k_2 : \pi^{-1}(U_2) \to K$. Since k_{12} is locally constant, Lemma A.3.9 implies that $\delta^l(k_i)(X_p)$ is the same for $i = 1, 2$ and thus

$$X_p \mapsto \delta^l(k_i)(X_p) \text{ if } p \in \pi^{-1}(U_i)$$

defines a connection 1-form on $\mathcal{P}_k$. Since the above identifications are obtained by evaluating $f \in C^\infty(P_k, \mathfrak{k})^K$ along a sections on which k_i is constant, this shows that the induced covariant derivative is in fact given by $f \mapsto df$. ■

Remark B.3.15 (Canonical flat connection). More generally, we call a smooth principal K-bundle $\mathcal{P}$ over M *flat* if one of the following equivalent conditions is satisfied

i) $\mathcal{P}$ has a smooth open trivialising system $(U_i, \sigma_i)_{i \in I}$ such that all corresponding transition functions $k_{ij} : U_i \cap U_j \to K$ are constant

ii) $\mathcal{P} \cong \mathcal{P}_\varphi$, where $\varphi : \pi_1(M) \to K$ is a homomorphism and $P_\varphi = \widetilde{M} \times K/\sim$ with $(m, k) \sim (m \cdot d, \varphi(d)^{-1} \cdot k)$ and canonical bundle projection and K-action. Here $\widetilde{M}$ denotes the universal covering of M, on which $\pi_1(M)$ acts canonically from the right.

In the case of a flat bundle, we have a canonical (flat) connection, constructed as follows. The $T_e(K) \cong \mathfrak{k}$-valued Maurer–Cartan form κ_{MC}, $X_k \mapsto T_{\lambda_{k^{-1}}}(k)(X_k)$ on K induces a $\pi_1(M)$-invariant connection 1-form $\widetilde{A} := \mathrm{pr}_2^* \kappa_{MC}$ on $\widetilde{M} \times K$. Since the fibres of $\pi : \widetilde{M} \times K \to P_\varphi$ are discrete, $\widetilde{A}$ vanishes in particular on the tangent spaces of the fibres and thus is the pull-back of a $\mathfrak{k}$-valued 1-form $A \in \Omega^1(p_\varphi, \mathfrak{k})$, i.e., we have $\pi^* A = \widetilde{A}$. This implies immediately that A is a connection 1-from on $\mathcal{P}_\varphi$.

We now consider the covariant derivative corresponding to A for an associated vector bundle. Let $\lambda : K \times Y \to Y$ be a smooth action and let $\lambda(\mathcal{P})$ be the associated bundle. Then we may identify $C^\infty(P, Y)^\lambda$ with

$$C^\infty_\varphi(M, Y) := \{f \in C^\infty(\widetilde{M}, Y) : f(x \cdot d) = \lambda(\varphi(d)^{-1}, f(x))\}.$$

With respect to these identifications, the covariant derivative induced from A is

$$d^A : C^\infty_\varphi(M, Y) \to \Omega^1(M, \lambda(\mathcal{P})), \quad d^A\,\eta.X_p = d\eta.X_p,$$

where we identify $T_m M$ with $T_{\widetilde{m}}\widetilde{M}$ canonically. ■

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Notation

$C^n(U,Y)$	space of n-times differentiable maps	8
$C^\infty(U,Y)$	space of smooth maps	8
f_{int}	restriction of f to interior	8
$C^n(U,Y)$	space of n-times differentiable maps	8
$C^\infty(U,Y)$	space of smooth maps	8
$\mathcal{O}(U,Y)$	space of holomorphic maps	8
$d^n f$	higher differential	8
Y^+	intersection of half-spaces	9
$(U_i,\varphi_i)_{i\in I}$	differential structure	9
$\mathrm{int}(M)$	interior of a manifold with corners	10
$\partial(M)$	boundary of a manifold with corners	10
$C^n(M,N)$	set of n-times differentiable maps	10
$C^\infty(M,N)$	set of smooth maps	10
$\mathcal{O}(M,N)$	set of holomorphic maps	10
T_mM	tangent space	11
TM	tangent bundle	11
$Tf : TM \to TN$	tangent map	11
T^nM	higher tangent bundle	11
$T^mf : T^mM \to T^mN$	higher tangent map	11
$\lfloor C,W \rfloor$	basic open set in c.-o. topology	12
$C(X,Y)_c$	space of maps with c.-o. topology	12
C^∞-topology	topology on space of smooth maps	13
$\Omega^p(M,\mathcal{E})$	space of p-forms with values in vector bundle	13
res	restriction map (for sections in vector bundles)	14
$S_{\overline{\mathfrak{U}}}(\mathcal{E})$	space of restricted sections	15

glue	gluing map (for sections in vector bundles)	15
$S_{\mathfrak{U}}(\mathcal{E})$	space of restricted sections	15
res	restriction map (for group valued functions)	19
$G_{\overline{\mathfrak{V}}}$	space of restricted maps	19
glue	gluing map (for group valued functions)	19
$\operatorname{Aut}_c(\mathcal{P})$	group of continuous bundle automorphisms	28
$\operatorname{Gau}_c(\mathcal{P})$	group of continuous gauge transformations	28
$\operatorname{Aut}(\mathcal{P})$	group of smooth bundle automorphisms	28
F_M	map induced on base by $F \in \operatorname{Aut}(\mathcal{P})$	28
$\operatorname{Diff}(M)_{\mathcal{P}}$	image of the homomorphism $Q : \operatorname{Aut}(\mathcal{P}) \to \operatorname{Diff}(M)$	28
$\operatorname{Gau}(\mathcal{P})$	gauge group (group of smooth gauge transformation)	28
$C^{\infty}(P, K)^K$	group of K-equivariant smooth maps	28
$\mathfrak{gau}(\mathcal{P}) = C^{\infty}(P, \mathfrak{k})^K$	gauge algebra (algebra of K-equivariant maps)	29
$\mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})$	gauge algebra in local coordinates	29
$\mathfrak{g}_{\mathcal{V}}(\mathcal{P})$	gauge algebra in local coordinates	29
$G_{\overline{\mathcal{V}}}(\mathcal{P})$	gauge group in local coordinates	31
$\varphi_* : U \subseteq G_{\overline{\mathcal{V}}}(\mathcal{P}) \to \mathfrak{g}_{\overline{\mathcal{V}}}(\mathcal{P})$	chart for the gauge group in local coordinates	32
$C(P, K)^K$	group of K-equivariant continuous maps	37
$G_{c,\overline{\mathcal{V}}}(\mathcal{P})$	continuous gauge group in local coordinates	38
$G_{c,\mathcal{V}}(\mathcal{P})$	continuous gauge group in local coordinates	38
$C^n(\mathcal{U}, A)$	n-cochains	57
∂_n	boundary operator on cochains	57
$\check{H}^n_c(M, A)$	continuous abelian Čech cohomology	57
$\check{H}^n_s(M, A)$	smooth abelian Čech cohomology	57
$\check{H}^n_c(M, K)$	continuous non-abelian Čech cohomology	57
$\check{H}^n_s(M, K)$	smooth non-abelian Čech cohomology	57
$\mathrm{U}(\mathcal{H})$	unitary group	58

$\mathrm{PU}(\mathcal{H})$	projective unitary group	58
$K(n,\pi_n(X))$	Eilenberg–MacLane space	58
$K^0(M)$	K-theory of M	58
$\mathrm{Fred}(\mathcal{H})$	Fredholm operators of $\mathcal{H}$	59
$K_{\mathcal{P}}(M)$	twisted K-theory of M	59
$\widetilde{g}$	lift of $g \in \mathrm{Diff}(M)$ to bundle automorphism	61
$\mathrm{Exp} : TM \to M$	exponential mapping of Riemannian metric	62
$S : O \to \mathrm{Aut}(\mathcal{P})$	section of $\mathrm{Aut}(\mathcal{P}) \ni F \mapsto F_M \in \mathrm{Diff}(M)$	63
(T,ω)	smooth factor system	67
$\mathrm{ev} : C(P,K)^K \to K$	evaluation fibration	76
$C_*(P,K)^K$	pointed gauge group	76
$C_A(X,Y)$	continuous maps with $f(A)=\{*\}$	78
$\delta_n : \pi_n(B) \to \pi_{n-1}(F)$	n-th connecting homomorphism	85
$\alpha\#\beta$	commutator map defining $\langle\cdot,\cdot\rangle_S$	86
$\langle a,b\rangle_S$	Samelson product	87
$\alpha\diamond\beta$	map defining $\langle\cdot,\cdot\rangle_{WH}$	89
$\langle\cdot,\cdot\rangle_{WH}$	Whitehead product	89
$\mathbb{P}\mathbb{H}^{n-1}$	projective space	91
$\pi_n^{\mathbb{Q}}(G) := \pi_n(G)\otimes\mathbb{Q}$	rational homotopy groups	92
$\mathfrak{z}_M(Y)$	target space for the covariant cocycle on $\mathfrak{gau}(\mathcal{P})$	96
$\lambda_\alpha(\omega)$	integral of $\omega \in \Omega^1(M,Y)$ over α	96
$\omega_{\kappa,A}$	continuous cocycle on $\mathfrak{gau}(\mathcal{P})$	96
$\widehat{\mathfrak{gau}(\mathcal{P})}$	central extension of the gauge algebra	97
$\mathrm{per}_\omega : \pi_2(G) \to \mathfrak{z}$	period homomorphism	98
$\kappa_{KC} : \mathfrak{k}\times\mathfrak{k} \to \mathbb{R}$	Cartan–Killing form	100
$\Pi_{\omega_K} = \mathrm{im}(\mathrm{per}_{\omega_K})$	period group of covariant cocycle	102
f_{Gau}	map induced by pull-backs	103
$f_{\mathfrak{gau}}$	map induced by pull-backs	103
$r_A : \mathrm{Aut}(\mathcal{P}) \to \Omega^1(M,\mathrm{Ad}(\mathcal{P}))$	cocycle for action of $\mathrm{Aut}(\mathcal{P})$ on $\Omega^1(M,\mathrm{Ad}(\mathcal{P}))$	111
$\widehat{\mathrm{Gau}(\mathcal{P})}_0$	central extension of gauge group	113
$C_k^\infty(\mathbb{S}^1,K)$	twisted loop group	116

$\mathfrak{g}_k := \widehat{C_k^\infty(\mathbb{S}^1, \mathfrak{k})}$	affine Kac–Moody algebra	118
$G_k := \widehat{C_k^\infty(\mathbb{S}^1, K)}$	affine Kac–Moody group	118
$H^n(M, \lambda_0(\mathcal{P}))$	twisted cohomology	123
$\exp_G : \mathfrak{g} \to G$	exponential function of G	127
$\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$	central extension of Lie algebras	129
$\widehat{\mathfrak{g}}_\omega$	central extension given by cocycle ω	130
$H_c^2(\mathfrak{g}, \mathfrak{z})$	second continuous Lie algebra cohomology	130
$Z \hookrightarrow \widehat{G} \twoheadrightarrow G$	central extension of Lie groups	130
$\widehat{G}_f$	central extension given by cocycle f	131
$H_s^2(G, Z)$	second smooth Lie group cohomology	131
$\lambda_g : M \to M$	$\lambda(g, \cdot)$ for a smooth action $\lambda : G \times M \to M$	132
$\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$	adjoint action	132
$\lambda_m : G \to M$	orbit map $\lambda(\cdot, m)$ for a smooth action $\lambda : G \times M \to M$	133
$S_c(\mathcal{E})$	space of continuous sections in vector bundle $\mathcal{E}$	137
$\rho_k : P \to P$	$\rho(\cdot, k)$ for action $\rho : P \times K \to P$	138
$k_\sigma : \pi^{-1}(U) \to K$	map with $\sigma(\pi(p)) \cdot k_\sigma(p) = p$ for section $\sigma : U \to P$	139
$\mathcal{U} = (U_i, \sigma_i)_{i \in I}$	open trivialising system	139
$\overline{\mathcal{U}} = (\overline{U}_i, \sigma_i)_{i \in I}$	closed trivialising system	139
$\mathcal{K} = (k_{ij} : U_i \cap U_j \to K)_{i,j \in I}$	cocycle	140
$\mathrm{Bun}(X, K)$	equivalence classes of principal K-bundles over X	142
$\lambda(\mathcal{P})$	bundle associated to smooth action λ	143
$f^*(\mathcal{P})$	pull-back bundle	144
$f_{\mathcal{P}}$	map induced on pull-back bundle	144
A_{2g}	bouquet of $2g$ circles	147
$\mathcal{V}_{\mathrm{vert}}(P)^K$	vertical K-invariant vector fields	149
$\tau_p : \mathfrak{k} \to T_pP$	derivative of the orbit map	149
$\rho_p : K \to P$	orbit map $\rho(p, \cdot)$ for action $\rho : P \times K \to K$	149
$\nabla : \Omega^0(M, \mathcal{E}) \to \Omega^1(M, \mathcal{E})$	covariant derivative	151

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