# A bundle-theoretic perspective to Kac–Moody groups

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May 31, 2007

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## Outline

Kac–Moody algebras and groups

Central extensions and cocycles

Cocycles for twisted loop algebras and -groups

Computation of homotopy groups

# Kac-Moody algebras

Generalisations of simple, finite-dimensional, complex Lie algebras:

- $A \in Mat_n(\mathbb{Z})$  : generalised Cartan matrix
- ▶ g(A) : assoc. Kac–Moody algebra (generators and relations)

- $\mathfrak{g}(A)$  finite-dimensional  $\Leftrightarrow A$  is positive definite
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- $\mathfrak{g}(A)$  affine : $\Leftrightarrow A$  positive semidefinite and  $\operatorname{rk}(A) = n 1$ Realisation in the affine case:
  - $\mathfrak{k} := \mathfrak{g}(\overline{A})$  finite-dimensional, simple  $(\overline{A}: \text{ generic from } A)$
  - $L\mathfrak{k} := \mathbb{C}[t, t^{-1}] \otimes \mathfrak{k}$
  - $\sigma \in Aut(\mathfrak{k})$  with  $\sigma^m = \mathbb{1}$  for  $m \in \{1, 2, 3\}$
  - ▶ Set  $\varepsilon := \exp(\frac{2\pi i}{m}) \rightsquigarrow \widetilde{\sigma} \in \operatorname{Aut}(L\mathfrak{k}), \ t^j \otimes x \mapsto \varepsilon^{-j} t^j \otimes \sigma(x).$

•  $L\mathfrak{k}_{\sigma} := (L\mathfrak{k})^{\sigma}$  (twisted loop algebra)

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#### Theorem (Kac)

 $L\mathfrak{k}_{\sigma} \oplus K\mathbb{C} \oplus d\mathbb{C}$  is a realisation of  $\mathfrak{g}(A)$  and  $\mathfrak{g}(A)' = L\mathfrak{k}_{\sigma} \oplus K\mathbb{C}$ , where K is the canonical central element and d acts as a derivation on  $L\mathfrak{k}_{\sigma} \oplus K\mathbb{C}$ .

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The Tits functor  $A \mapsto G(A)$  produces groups, associated to Kac-Moody algebras and a partially defined exponential function

$$x\mapsto \exp_{\mathcal{G}}(x) \ \ ext{for} \ \ x\in igcup_lpha \mathfrak{g}_lpha$$

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#### Proposition

Let  $(V, \pi)$  be an integrable  $\mathfrak{g}(A)'$ -module. Then there exists a unique homomorphism  $\widetilde{\pi} : G(A) \to \operatorname{Aut}(V)$  satisfying

$$\exp \circ \pi(x) = \widetilde{\pi} \circ \exp_G(x)$$
 for  $x \in \bigcup_{lpha} \mathfrak{g}_{lpha}$ 

This fact should (at least) be satisfied by each notion of a Kac–Moody group.

## Twisted loop algebras and algebras of sections

- switch form the complex to the real case
- $\mathfrak{k}$  : finite-dimensional Lie algebra,  $\varphi \in \mathsf{Aut}(\mathfrak{k})$
- $P_{\varphi}$ : Lie algebra-bundle over  $\mathbb{S}^1$  with holonomy  $\varphi$ , i.e.

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algebra of sections:

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#### Problem

Construct central extensions  $\widehat{\Gamma P_{\varphi}}$  and corresponding Lie groups  $G(\widehat{\Gamma P_{\varphi}})$ , satisfying the previous integrability condition.

## Differential calculus in locally convex spaces

#### Definition (Milnor)

X, Y: locally convex spaces,  $U \subseteq X$  open subset. A map  $f: U \rightarrow Y$  is differentiable, if the differential quotients

$$df(x,v) := \lim_{h \to 0} \frac{f(x+hv) - f(x)}{h}$$

exist for each  $v \in X$ , and  $(x, v) \mapsto df(x, v)$  is jointly continuous.

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## Example

- unit groups of Banach algebras
- mapping groups  $C^{\infty}(M, G)$ , sections in Lie group bundles  $\Gamma \mathcal{K}$
- ▶ groups of diffeomorphisms Diff(M) for compact M

# Cohomology groups for central extensions

Setting:

- ▶ 3: locally convex space,
- $\Gamma \subseteq \mathfrak{z}$ : discrete subgroup  $\rightsquigarrow Z := \mathfrak{z}/\Gamma$  with  $\pi_1(Z) \cong \Gamma$
- G: connected l.c. Lie group with Lie algebra  $\mathfrak{g} := L(G)$ .

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Lie algebra cocycle:

 $\omega:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{z},$  continuous, bilinear, alternating, satisfying

$$\omega([x,y],z) + \omega([y,z],x) + \omega([z,x],y) = 0$$

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- $\rightsquigarrow$  cohomology groups  $H^2_c(\mathfrak{g},\mathfrak{z}), H^2_s(G,Z)$
- $\stackrel{\rightsquigarrow}{\longrightarrow} D: H^2_s(G, Z) \mapsto H^2_c(\mathfrak{g}, \mathfrak{z}), \ [f] \mapsto [(d^2f(e, e))_{\mathsf{asym}}]$ is well-defined

#### Central Extensions and smooth Lie group cohomology

Central extension of Lie algebras: linearly split exact sequence

$$0 \to \mathfrak{z} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

Central extension of Lie groups: locally trivial exact sequence

$$1 \rightarrow Z \rightarrow \widehat{G} \rightarrow G \rightarrow 1$$

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Theorem (Neeb (02))

With the canonical identifications  $H^2_s(G, Z) \cong \operatorname{Ext}_{Lie}(G, Z)$  and  $H^2_c(\mathfrak{g}, \mathfrak{z}) \cong \operatorname{Ext}_c(\mathfrak{g}, \mathfrak{z})$  there is an exact sequence

$$\operatorname{Hom}(\pi_1(G), Z) \to \operatorname{Ext}_{Lie}(G, Z) \xrightarrow{D} \\ \operatorname{Ext}_c(\mathfrak{g}, \mathfrak{z}) \xrightarrow{P} \operatorname{Hom}(\pi_2(G), Z) \oplus \operatorname{Hom}(\pi_1(G), \operatorname{Lin}(\mathfrak{g}, \mathfrak{z})),$$

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⇒ the analysis of  $P_1([\omega])$  is crucial for the integrability of a cocycle (resp. a central extension)

▶ £: finite-dimensional Lie algebra

- $\kappa : \mathfrak{k} \times \mathfrak{k} \to V$  biliniear, symmetric,  $\kappa([x, y], z) = \kappa(x, [y, z])$

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Theorem (Pressley, Segal (86); Maier, Neeb (03))

If  $\kappa$  is universal, then  $P([\omega_{\kappa}])$  vanishes, i.e., the corresponding central extension of  $C^{\infty}(M, \mathfrak{k})$  integrates to a central extension of  $C^{\infty}(M, K)_0$ . Moreover, both central extensions are universal.

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- ▶  $\pi_0(K)$ -module  $(V, \rho)$  ( $\Leftrightarrow$  K-module with  $K_0 \subseteq \ker(\rho)$ )
- $\kappa: \mathfrak{k} \times \mathfrak{k} \to V$  bilinear, symmetric, *K*-equivariant
- ▶  $\nabla$ : connection on *P* (and on associated vector bundles)
- $\rightsquigarrow$  associated vector bundles  $\mathfrak{K}:= P \times_{\mathsf{Ad}} \mathfrak{k}$  and  $\mathbb{V}:= P_0 \times_\rho V$
- $\rightsquigarrow$  induced maps between sections  $\Gamma \kappa : \Gamma \mathfrak{K} \times \Gamma \mathfrak{K} \to \Gamma \mathbb{V}$

• 
$$\mathfrak{g} := \Gamma \mathfrak{K}, \ \mathfrak{z} := \Omega^1(M, \mathbb{V})/d(\Gamma P_{\mathbb{V}})$$

 $\stackrel{\rightsquigarrow}{\longrightarrow} \widetilde{\omega_{\kappa,\nabla}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z} \quad (f,g) \mapsto [\Gamma \kappa(f,\nabla g)] \text{ is a Lie algebra} \\ \text{cocycle}$ 

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#### Proposition

Different choices of connections on P yield equivalent cocycles.

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#### Proposition

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#### Question

Is this cocycle universal, e.g., if ℓ is assumed to be simple? (This is true for trivial bundles!)

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## Theorem (W. (07))

The group of sections  $\Gamma \mathcal{K}$  in the Lie group bundle  $\mathcal{K} := P \times_{conj} K$  ("gauge group") is an infinite-dimensional Lie group with Lie algebra  $L(\Gamma \mathcal{K}) = \Gamma \mathfrak{K}$ .

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#### Theorem (Neeb, W. (07))

If  $\kappa$  is universal,  $\pi_0(K)$  is finite and acts trivially on  $H_1(P_0)$ , then  $P([\widetilde{\omega_{\kappa}}])$  vanishes, i.e., the corresponding central extension of  $\Gamma \mathfrak{K}$  integrates to a central extension of  $(\Gamma \mathcal{K})_0$ .

# Further Generalisations

- approach generalises to algebras/group of sections in Lie algebra- and Lie group bundles
- ► additional ingredient: smooth action \(\rho\) : \(K \times H \rightarrow Lie group bundle \(\mathcal{H}\) and Lie algebra bundle \(\mathcal{S}\).
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#### Further Generalisations

- approach generalises to algebras/group of sections in Lie algebra- and Lie group bundles
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#### Example

- $K = \mathbb{T} \cong \mathbb{S}^1$ ,  $H = C^{\infty}(\mathbb{S}^1, G)$  for finite-dim. Lie group G
- ► action:  $\rho(t,\gamma)(t') = \gamma(t \cdot t') \Rightarrow \Gamma \mathcal{H} \cong C^{\infty}(P,G)$
- ► more generally, for N compact, Diff(N) acts canonically on  $C^{\infty}(N, G)$  and a Diff(N)-principal bundle yields an associated fibre bundle  $P_N = P \times_{\text{Diff}(N)} N$  with  $\Gamma \mathcal{H} \cong C^{\infty}(P_N, G)$

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#### Problem

Different connections do not lead to equivalent cocycles any more!

The route back to Kac–Moody groups

#### Theorem (Milnor (84))

If H is a regular Lie group, G is a simply connected Lie gorup, then each morphism  $\varphi : L(G) \rightarrow L(H)$  lifts to a unique morphism  $\Phi : G \rightarrow H$ .

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#### Corollary

- ▶ G simply connected, V Banach space
- ⇒ each representation  $\varphi : L(G) \rightarrow End(V)$  lifts to a unique representation  $\Phi : G \rightarrow GL(V)$

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#### Corollary

- ▶ G simply connected, V Banach space
- ⇒ each representation  $\varphi : L(G) \rightarrow End(V)$  lifts to a unique representation  $\Phi : G \rightarrow GL(V)$
- ~> calculate the fundamental group of the constructed central extension

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▶  $P_{\varphi}$  bundle with holonomy  $\varphi$ ,  $\Gamma P_{\varphi}$  twisted loop algebra

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Fibration  $ev_0: \Gamma P_\Phi o K$ ,  $\gamma \mapsto \gamma(0)$  gives a long exact sequence

$$\ldots \to \pi_1(\underbrace{\ker(\mathsf{ev}_0)}_{\simeq \Omega K}) \to \pi_1(\Gamma P_{\Phi}) \to \pi_1(K) \to \ldots$$

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 $\Rightarrow \pi_1(\Gamma P_\Phi)$  vanishes, as  $\pi_1(K)$  and  $\pi_2(K)$  do

• Setting as on the previous slide,  $\mathfrak{g} := \Gamma P_{\varphi}, \ G := (\Gamma P_{\Phi})_0$ 

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•  $\kappa : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$  Cartan–Killing form (universal, for  $\mathfrak{k}$  is *compact*)

 $\rightsquigarrow$  yields a cocycle  $\widetilde{\omega_\kappa}$  and a central extension

$$0 \to \mathfrak{z} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

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 $\rightsquigarrow$  integrates, by the above theorem, to a central extension

$$1 \rightarrow Z \rightarrow \widehat{G} \rightarrow G \rightarrow 1$$

for some  $Z = \mathfrak{z}/\Gamma$  and gives rise to an exact sequence

$$\ldots \rightarrow \pi_2(G) \xrightarrow{\delta} \pi_1(Z) \rightarrow \pi_1(\widehat{G}) \rightarrow \pi_1(G) \rightarrow \ldots$$

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 $\sim \rightarrow$  can choose  $\Gamma$  such that  $\delta$  is surjective  $\Rightarrow \pi_1(\widehat{G})$  vanishes

Lifting property for Kac–Moody groups

Proposition

For the cocycle

$$\Gamma P_{arphi} imes \Gamma P_{arphi} o \Omega^1(\mathbb{S}^1,\mathbb{V})/d(\Gamma\mathbb{V}), \ \ (f,g) \mapsto [\Gamma\kappa(f,
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- →  $\hat{G}$  has the lifting property for the Kac–Moody algebra  $\hat{\mathfrak{g}}$ → call  $\hat{G}$  the Kac–Moody group associated to  $\varphi \in \operatorname{Aut}(\mathfrak{k})$
- $\stackrel{\sim}{\longrightarrow} \text{ generalisations as central extensions of } \Gamma\mathfrak{K} \text{ for } \mathfrak{K} = P \times_{\mathsf{Ad}} \mathfrak{k} \text{ for } \mathfrak{flat principal bundles } P \text{ seems to be appropriate}$
- → access to Aut( $\Gamma$  $\Re$ ) as the group Aut(P) of bundle automorphisms (→ leads to Lie group structures on Aut( $\Gamma$  $\Re$ ))