# Gauge groups for lifting gerbes and principal 2-bundles

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# Motivation for considering principal 2-bundles

Problem: Find geometric models for cohomology Examples in differential topology

- $H^1(M, \mathbb{Z}_2)$ : classifies spin structures on a spin manifold M
- $H^2(M,\mathbb{Z})$ : classifies complex line bundles over M
- ► H<sup>3</sup>(M, Z): classifies P U(H)-bundles over M

#### Examples in group cohomology

- $H^1(G, A)$  classifies sections of  $A \rtimes G \to G$
- $H^2(G, A)$  classifies abelian extensions of G by A

Question: Is there a consistent procedure to construct these models? ~> prinicpal 2-bundles (categorified principal bundles)

### Outline

2-groups (categorified groups)

Principal 2-bundles and lifting gerbes

Gauge groups of lifting gerbes

Arbitrary Lie group bundles and central extensions

# From ordinary groups to 2-groups

Basic examples of groups: symmetric groups  $G := \text{Sym}(X) := \{f : X \to X | f \text{ bijective}\} \text{ for } X \text{ a set}$ 

Properties:

•  $\mu: G \times G \rightarrow G$  associative map (given by composition)

•  $1 \in G$  and existence of inverses

 $\rightsquigarrow$  leads to the abstract notion of a group

Categorified version of a symmetric group  $\mathcal{G} := \mathcal{S}ym(\mathcal{C}) := \{F : \mathcal{C} \to \mathcal{C} | F \text{ equivalence}\} \text{ for } \mathcal{C} \text{ a category}$ 

Properties:

▶ G is a category with natural equivalences between functors  $F \Rightarrow G$ 

•  $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$  associative functor (given by composition)

•  $1 \in \mathcal{G}$  and existence of inverses (by definition)

 $\rightsquigarrow$  leads to the abstract notion of a 2-group

# 2-groups

#### Definition

A 2-group is a small category  $\mathcal{G}$  with a functor  $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ , an object  $\mathbb{1} \in \mathcal{G}$  and (coherent) natural equivalences

$$\mu \circ (\mu \times \mathsf{id}_{\mathcal{G}}) \Rightarrow \mu \circ (\mathsf{id}_{\mathcal{G}} \times \mathcal{G})$$
$$\mu \circ (\mathbb{1} \times \mathsf{id}_{\mathcal{G}}) \Rightarrow \mathsf{id}_{\mathcal{G}} \Rightarrow \mu \circ (\mathsf{id}_{\mathcal{G}} \times \mathbb{1})$$

such that each morphism is invertible and for each  $g \in \text{Obj}(\mathcal{G})$  there exists an  $g^{-1}$  with  $\mu_0(g, g^{-1}) \cong \mathbb{1}$  and  $\mu_0(g^{-1}, g) \cong \mathbb{1}$ .

#### Remark

2-groups are monoidal categories and morphisms between them are monoidal functors.

2-groups are *no* 2-categories! Rather, in a 2-category, equivalences of objects are 2-groups (compare with categories and groups).

A (coherent) 2-group, in which all natural equivalences are the identity, is called a *strict* 2-group.

#### Examples of 2-groups

• C: category  $\Rightarrow Aut(G)$  is a 2-group (in general non-strict)

• X: topological space,  $x \in X \rightsquigarrow$  define  $\Pi_x(X)$ 

 $\operatorname{Obj}(\Pi_x(X)) := \Omega_x(X)$  $\operatorname{Hom}_{\Pi_x(X)}(f,g) := \{ \text{homotopies } F : f \Rightarrow g \}$ 

 $\Rightarrow$  is a (non-strict) 2-group with the obvious compositions

- G: group ⇒ (G, G) is a (discrete) strict 2-group with all structure maps id<sub>G</sub> and µ given by multiplication in G
- *H*: abelian group  $\Rightarrow$  (*H*, {\*}) is a strict 2-group
- β: H → G: crossed module ⇒ (G, H ⋊ G) is a strict 2-group as follows

## Crossed modules

#### Definition

A crossed module consists of a group homomorphism  $\beta: H \to G$ and an action  $G \to Aut(H)$  such that

$$\begin{split} \beta(g.h) &= g \cdot \beta(h) \cdot g^{-1} \quad (\text{equivariancy}) \\ \beta(h).h &= h \cdot h' \cdot h^{-1} \qquad (\text{Peiffer identity}) \end{split}$$

- Note:  $ker(\beta) \subseteq Z(H)$  follows from the Peiffer identity
  - $\operatorname{im}(\beta) \trianglelefteq G$  by equivariancy

#### Examples

- $\beta: H \rightarrow \{*\}$  for H abelian
- inclusion of a normal subgroup  $\beta : N \hookrightarrow G$  ( $\beta$  injective)
- central extension  $\beta : \widehat{G} \to G$  ( $\beta$  surjective)

#### Crossed modules yield 2-groups

From a crossed module  $\beta : H \to G$ , one constructs a 2-group  $\mathcal{G}_{\beta}$  as follows. Put  $Obj(\mathcal{G}_{\beta}) = G$ ,  $Mor(\mathcal{G}_{\beta}) = H \rtimes G$  and set

$$g = ullet \xrightarrow{(h,g)} ullet = eta(h) \cdot g$$

Define composition in  $\mathcal{G}_{\beta}$  by

$$\left( \bullet \xrightarrow{(h,g)} \bullet \xrightarrow{(h',\beta(h) \cdot g)} \bullet \right) \mapsto \left( \bullet \xrightarrow{(h' \cdot h,g)} \bullet \right)$$

and the multiplication functor  $\mu:\mathcal{G}_\beta\times\mathcal{G}_\beta\to\mathcal{G}_\beta$  by

$$\left(\bullet \xrightarrow{(h,g)} \bullet, \bullet \xrightarrow{(h',g')} \bullet\right) \mapsto \left(\bullet \xrightarrow{(h\cdot g.h',g\cdot g')} \bullet\right)$$

#### Theorem (folklore)

Each strict 2-group arises in this way from a crossed module

# Smooth spaces

#### Definition

A *smooth 2-space* is a small category C, s.th. spaces of objects, morphisms, composable tuples and triples of morphisms are smooth manifolds and all structure maps are smooth. A functor (natural transformation) is smooth, if the map it represents is so.

A 2-group is a *Lie 2-group* if it is a smooth 2-space and the functors (and natural equivalences) defining the group structure are smooth.

A crossed module  $\beta : H \to G$  is smooth if  $\beta$  and  $(g, h) \mapsto g.h$  are smooth maps.

#### Lemma

 $\beta: H \to G$  a smooth crossed module  $\Rightarrow \mathcal{G}_{\beta}$  is a Lie 2-group.

 $\leadsto$  take smooth crossed modules as the working definition of a Lie 2-group.

## Principal 2-bundles

Fix a Lie 2-group  $\mathcal{G} := \mathcal{G}_{\beta}$  and a smooth 2-space  $\mathcal{M} = (M, M)$  with only identity morphisms (for M a smooth manifold).

#### Definition

A strict  $\mathcal{G}\text{-}2\text{-space}$  is a smooth 2-space  $\mathcal{P}$  and a smooth functor  $\rho:\mathcal{P}\times\mathcal{G}\to\mathcal{P}$  such that

$$\rho \circ (\rho \times \operatorname{id}_{\mathcal{G}}) = \rho \circ (\operatorname{id}_{\mathcal{P}} \times \mu) \quad \text{and} \quad \rho \circ (\operatorname{id}_{\mathcal{P}} \times \mathbb{1}) = \operatorname{id}_{\mathcal{P}}$$

(equality of functors). Similarly, one defines (strict) morphisms between strict  $\mathcal{G}\mbox{-}2\mbox{-spaces}.$ 

A principal  $\mathcal{G}$ -2-bundle over  $\mathcal{M}$  is a strict  $\mathcal{G}$ -2-space  $\mathcal{P}$ , together with a smooth morphism  $\mathcal{P} \to \mathcal{M}$ , s.th.  $\mathcal{P}$  is locally trivial over  $\mathcal{M}$ . A bundle morphism is a morphism  $F : \mathcal{P} \to \mathcal{P}'$  of  $\mathcal{G}$ -2-spaces satisfying  $\pi' \circ F = \pi$ .

## Example: Lifting gerbes

Ingredients: •  $P \rightarrow M$  an ordinary principal *G*-bundle •  $\beta : \widehat{G} \rightarrow G$  a central extension of *G* 

 $\rightsquigarrow$  yields a principal  $\mathcal{G}_{\beta}\mbox{-}2\mbox{-}bundle$   $\mathcal P$  by setting

- $\operatorname{Obj}(\mathcal{P}) = P$ ,  $\operatorname{Hom}(p, p') = \{p \xrightarrow{(p, p', \hat{g})} p' : p = p' \cdot \beta(\hat{g})\}$
- ▶ composition in *P*:

$$\left(p \xrightarrow{(p,p',\hat{g})} p' \xrightarrow{(p',p'',\hat{g}')} p''\right) \mapsto \left(p \xrightarrow{(p,p'',\hat{g},\hat{g}')} p''\right)$$

• action of  $\mathcal{G}_{\beta}$ :

$$\underbrace{(p,g) \mapsto p \cdot g}_{\text{on objects}} \quad \underbrace{(p,p',\hat{g}), (\hat{g}',g) \mapsto (p \cdot g, p' \cdot (\beta(\hat{g}')g), g^{-1}.(\hat{g}\hat{g}'))}_{\text{on morphisms}}$$

 $\rightsquigarrow$  local trivialisations of *P* yield local trivialisations of *P*.

# Classification of principal 2-bundles

 $\begin{array}{l} \mbox{Definition} \\ F: \mathcal{P} \rightarrow \mathcal{P}' \mbox{ a bundle equivalence } :\Leftrightarrow \\ & ex. \ F': \mathcal{P}' \rightarrow \mathcal{P} \mbox{ with } F \circ F' \cong id_{\mathcal{P}'} \mbox{ and } F' \circ F \cong id_{\mathcal{P}}. \\ & F \mbox{ is strict } :\Leftrightarrow F \circ F' = id_{\mathcal{P}'} \mbox{ and } F' \circ F = id_{\mathcal{P}} \mbox{ on the nose.} \end{array}$ 

## Theorem (folklore, Bartels '06, W.'08)

Principal  $\mathcal{G}_{\beta}$ -2-bundles over  $\mathcal{M}$  are calssified (up to strict equivalence) by  $H^2(\mathcal{M}, \mathcal{G}_{\beta})$ .

#### Remark

- $H^2(\mathcal{M}, \mathcal{G}_\beta)$  is the "well-known" non-abelian Čech cohomology.
- $\blacktriangleright \ \beta : \{*\} \to G \Rightarrow H^2(\mathcal{M}, \mathcal{G}_\beta) \cong \check{H}^1(M, G)$
- $\blacktriangleright \ \beta : H \to \{*\} \Rightarrow H^2(\mathcal{M}, \mathcal{G}_\beta) \cong H^2(M, A)$
- ▶  $\beta : H \to G$  centr. ext  $\Rightarrow H^2(\mathcal{M}, \mathcal{G}_\beta) \cong H^2(M, \ker(\beta))$
- higher n-groups yield geometric models for higher cohomolgy

# Gauge groups

Philosophy: Automorphism groups of geometric structures (principal bundles) give rise to interesting ( $\infty$ -dim.) Lie groups.

#### Reminder

 $P \to M$ : principal G-bundle  $\Rightarrow C^{\infty}(P,G)^G \cong \operatorname{Gau}(P)$  by the map

$$\mathcal{C}^\infty(\mathsf{P},\mathsf{G})^\mathsf{G} 
i \gamma \mapsto (\mathsf{p} \mapsto \mathsf{p} \cdot \gamma(\mathsf{p})) \in \mathsf{Gau}(\mathsf{P})$$

(note:  $p \cdot g \cdot \gamma(p \cdot g) = p \cdot g \cdot g^{-1} \cdot \gamma(p) \cdot g = p \cdot \gamma(p) \cdot g$ )

Theorem (Michor et.al. 90's, W. '07)  $C^{\infty}(P, G)^{G}$  is an infinite-dimensional Lie group, modelled on  $C^{\infty}(P, \mathfrak{g})^{G}$  (if M is compact and exp :  $\mathfrak{g} \to G$  is a local difform.).

# Gauge 2-groups

Philosophy: Automorphism groups of principal 2-bundles give rise to interesting  $\infty\text{-dim}.$  Lie 2-groups.

#### Lemma & Definition

 $\begin{aligned} \mathcal{P}: \text{ principal } \mathcal{G}\text{-2-bundle} \Rightarrow \mathcal{G}au(\mathcal{P}) &:= \{F: \mathcal{P} \rightarrow \mathcal{P}: F \text{ equiv.}\} \\ \text{ is a 2-group, the gauge 2-group of } \mathcal{P}. \end{aligned}$ 

#### Proposition

 $\mathcal{P} \to \mathcal{M}$ : principal  $\mathcal{G}$ -2-bundle  $\Rightarrow \mathcal{G}au(\mathcal{P}) \cong \mathcal{C}^{\infty}(\mathcal{P}, \mathcal{G})^{\mathcal{G}}$ 

In particular,  $Gau(\mathcal{P})$  is strict.

Question: Lie 2-group structures on  $\mathcal{C}^{\infty}(\mathcal{P},\mathcal{G})^{\mathcal{G}}$ ?

 $\mathcal{C}^{\infty}(\mathcal{P},\mathcal{G})^{\mathcal{G}}$  as Lie 2-group

Fix  $eta: H 
ightarrow {\sf G}$  (thus  ${\cal G}:={\cal G}_eta)$  and  ${\cal P}$ 

- ▶ have a functor L: (Lie 2-groups)<sub>str</sub> → (Lie 2-algebras)<sub>str</sub>
- $\mathcal{G}$  acts strictly on  $L(\mathcal{G})$  by  $\mathcal{A}d: \mathcal{G} \to \mathcal{A}ut(L(\mathcal{G}))$

► For 
$$\Gamma : \mathcal{P} \to \mathcal{G}$$
 in  $\mathcal{C}^{\infty}(\mathcal{P}, \mathcal{G})_0^{\mathcal{G}} := \operatorname{Obj}(\mathcal{C}^{\infty}(\mathcal{P}, \mathcal{G})^{\mathcal{G}})$ , one has  
 $\Gamma_0 \in \mathcal{C}^{\infty}(\mathcal{P}_0, \mathcal{G}_0)^{\mathcal{G}_0}$  (map on objects)  
 $\Gamma_1 \in \mathcal{C}^{\infty}(\mathcal{P}_1, \mathcal{G}_1)^{\mathcal{G}_1}$  (map on morphisms)

 $\Rightarrow \mathcal{C}^{\infty}(\mathcal{P},\mathcal{G})_{0}^{\mathcal{G}} \leq C^{\infty}(\mathcal{P}_{0},\mathcal{G}_{0})^{\mathcal{G}_{0}} \times C^{\infty}(\mathcal{P}_{1},\mathcal{G}_{1})^{\mathcal{G}_{1}} \text{ (closed)}$ 

- ▶ similar argument shows  $Mor(C^{\infty}(\mathcal{P},\mathcal{G})^{\mathcal{G}}) \cong C^{\infty}(\mathcal{P}_0,H)^{\mathcal{G}}$
- $C^{\infty}(\mathcal{P}_0, H)^G$  is Lie group, modelled on  $C^{\infty}(\mathcal{P}_0, L(H))^G$

Theorem (W. '08)

M compact and  $exp_H$ ,  $exp_H$  local difform.

 $\Rightarrow \mathcal{C}^{\infty}(\mathcal{P},\mathcal{G})^{\mathcal{G}} \text{ is strict Lie 2-group, modelled on } \mathcal{C}^{\infty}(\mathcal{P},L(\mathcal{G}))^{\mathcal{G}}$ 

# Lifting gerbes

Ingredients: •  $P \rightarrow M$  an ordinary principal *G*-bundle •  $Z \rightarrow \widehat{G} \xrightarrow{\beta} G$  a central extension of *G* 

Thus the Lie 2-group  $\mathcal{C}^{\infty}(\mathcal{P},\mathcal{G})^{\mathcal{G}}$  is associated to the push-forward crossed module

$$C^{\infty}(P,\widehat{G})^{G} \xrightarrow{\beta_{*}} C^{\infty}(P,G)^{G} \times C^{\infty}(M,Z)$$

(note:  $C^{\infty}(P,\widehat{G})^{G}$  is the group of sections in  $P \times_{G} \widehat{G}$ )

# Groups of sections in Lie group bundles

#### Ingredients: • $P \rightarrow M$ an ordinary principal *G*-bundle • $G \rightarrow Aut(H)$ an arbitrary smooth action

$$\label{eq:Gamma} \begin{split} \mathsf{\Gamma}\mathcal{H} &:= C^\infty(P,H)^G \text{ (group of sections of } \mathcal{H} := P \times_G H \text{)} \\ & \text{ is a Lie group, modelled on } \end{split}$$

 $\Gamma\mathfrak{h} := C^{\infty}(P, L(H))^{\mathcal{G}}$  (algebra of sections of  $\mathfrak{H} := P \times_{\mathcal{G}} \mathfrak{h}$ )

#### Questions:

- ► criteria to view G → Aut(H) as crossed module (e.g., H fin.-dim. semi-simple and G connected)
- ➤ ∞-dim. Lie theory for C<sup>∞</sup>(P, H)<sup>G</sup> (i.part., extension theory, central extensions)

#### In the sequel:

Construct central extensions of  $C^{\infty}(P, H)^{G}$  from the "canonical" ones of  $C^{\infty}(P, L(H))^{G}$ , using the machinery (Neeb '02).

# Central extensions of groups of section

- $P_0 := P/G_0$  squeezed  $\pi_0(G)$ -bundle (regular cover of M)
- ▶  $\pi_0(G)$ -module  $(V, \rho)$  ( $\Leftrightarrow$  *G*-module with  $G_0 \subseteq \ker(\rho)$ )
- $\kappa : \mathfrak{h} \times \mathfrak{h} \to V$  bilinear, symmetric, *G*-equivariant
- ►  $\nabla$ : connection on *P* (and on associated vector bundles)  $\rightsquigarrow$  associated vector bundle  $\mathbb{V} := P_0 \times_{\rho} V$
- $\rightsquigarrow$  induced maps between sections  $\Gamma \kappa : \Gamma \mathfrak{H} \times \Gamma \mathfrak{H} \to \Gamma \mathbb{V}$

▶ 
$$\mathfrak{h} := \Gamma \mathfrak{H}, \quad \mathfrak{z} := \Omega^1(M, \mathbb{V})/d(\Gamma \mathbb{V})$$
  
 $\rightsquigarrow \omega := \widetilde{\omega_{\kappa, \nabla}} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{z} \quad (f, g) \mapsto [\Gamma \kappa(f, \nabla g)] \text{ is a cocycle}$ 

 $\rightsquigarrow$  central extension  $\mathfrak{z} \hookrightarrow \mathfrak{z} \oplus_{\omega} \mathfrak{h} \twoheadrightarrow \mathfrak{h}$ 

#### Theorem (Neeb, W. '07)

If  $\kappa$  is universal and  $\pi_0(G)$  is finite, then the extension  $\mathfrak{z} \hookrightarrow \mathfrak{z} \oplus_{\omega} \mathfrak{h} \twoheadrightarrow \mathfrak{h}$  integrates (up to a curvature condition) to a central extension of  $(\Gamma \mathcal{H})_0$ .

- Curvature cond. is satisfied, e.g., if dim $(H) \leq \infty$
- Is this central extension universal, e.g., if h is simple?

## Example: Locally trivial bundles and Mapping groups

 $P \rightarrow M$ : locally trivial bundle with compact fibre

• consider  $\widetilde{P} := \bigcup_{x \in M} \operatorname{Diff}(P_x, N)$ .

 $\Rightarrow \widetilde{P}$  is a Diff(N) principal bundle with  $P = \widetilde{P} \times_{\text{Diff}(N)} \text{Diff}(N)$ 

- For C<sup>∞</sup>(N, K) for K arbitrary and consider the smooth automorphisc Diff(N)-action (φ, f) → f ∘ φ<sup>-1</sup>
- $\Rightarrow \mathcal{H} := \widetilde{P} \times_{\mathsf{Diff}(N)} C^{\infty}(N, K) \text{ is a Lie group bundle.}$

#### Proposition

$$\Gamma \mathcal{H} := C^{\infty}(\widetilde{P}, C^{\infty}(N, K))^{\mathrm{Diff}(N)} \cong C^{\infty}(P, K).$$

Remark

- Example is contrary to crossed modules
- integration theory for central extensions breaks down (integrality conditions cannot be checked easily, produces manifest counterexamples)

# Summary

- introduction to (Lie) 2-groups from elementary examples
- principal 2-bundles are classified by non-abelian (higher) Čech cohomology (ex.: lifting gerbes)
- gauge groups of principal 2-bundles are Lie 2-groups
- central extensions for groups of sections
- for details, see preprints on www.wockel.eu

# Thank you!